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Engineering Mechanics Qualifying Exam: Columbia University, Fall 2000

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Chapter 1

Continuum Mechanics

1.1 Elasticity

1.1.1 References

- Foundations of Solid Mechanics [8]

- Introduction to Continuum Mechanics [14]

- Elasticity, Prof. Gerard Ateshian, Columbia University, E6422, Fall 1999

- Elasticity, Prof. Rene B. Testa, Columbia University, E6315, Spring 2000

1.1.2 Glossary

Gibs Notation: A vector notation which is invariant of coordinate systems.

Summation Convention: the repetition of an index (whether superscript or subscript) in a term denotes a summation with respect to that index over its range.

The summation convention can also be used for differentiation.

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{1.1}$$

Range: The range of an index i is the set of n integer values 1 to n.

Dummy Index: an index which is summed over.

Free Index: an index which is not summed over.

Kronecker Delta: $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Permutation Symbol: $e_{ijk} = \begin{cases} 1 & even \\ -1 & odd \\ 0 & neither \end{cases}$

The permutation symbol and the Kronecker Delta are related as follows.

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{1.2}$$

Cartesian Tensors: Covariance and contravariance are equal. The transformation tensor for cartesian vectors is an orthogonal tensor Q. Orthogonality preserves the lengths and angles of the transformed vectors. Reflection, rotation, and translation are orthogonal translations. Also $Q^T Q = I$

Contraction: The process of equating and summing a covariant and contravariant index of a mixed tensor. the result of contraction is another tensor, if no free index is left the resulting quantity is a scalar.

$$\frac{\partial a^\beta}{\partial a^\alpha} = \delta^\beta_\alpha \tag{1.3}$$

Quotient Rule: The Quotient Rule can be used to determine if a function is a tensor without determining the law of transformation directly.

   Theorem 1: If $[A(i_1, i_2, i_3, \ldots, i_r)]$ is a set of functions of the variables x^i, and if the product $A(\alpha, i_2, i_3, \ldots, i_r)\xi^\alpha$ with an arbitrary vector ξ^α be a tensor of the type $A^{j_1 \ldots j_p}_{k_1 \ldots k_q}(x)$, then the set $A(i_1, i_2, i_3, \ldots, i_r)$ represents a vector of the type $A^{j_1 \ldots j_p}_{\alpha k_1 \ldots k_q}(x)$.

   Theorem 2: Similarly, if the product of a set of n^2 functions $A(\alpha, j)$ with an arbitrary tensor $B_{\alpha k}$ (and summed over α) is a covariant tensor of rank 2, then $A(i, j)$ represents a tensor of the type A^i_j.

Directional Vector: Given surfaces α_i of constant value where the functions $\{\alpha^1, \alpha^2, \ldots, \alpha^n\}$ can be written as functions of cartesian coordinates $\{x, y, z, \ldots\}$ (where n is the dimension of the system) and the jacobian is not equal to zero $J = |\nabla \vec{\alpha}| \neq 0$. Then covarient and contravarient directional vectors respectively are:

$$\frac{\partial x_i}{\partial \alpha_j}\hat{i}_i = \vec{g}_j \quad \text{and} \quad \frac{\partial \alpha_j}{\partial x_i}\hat{i}_j = \vec{g}^j \tag{1.4}$$

also the covariant derivative

$$\nabla = \vec{g}_i \frac{\partial}{\partial \alpha^i} \tag{1.5}$$

Physical Components of a Vector: The base vectors g_r and g^r are in general not unit vectors. Their lengths are:

$$|\mathbf{g}_r| = \sqrt{g_{rr}}, \quad |\mathbf{g}^r| = \sqrt{g^{rr}}, \quad \text{r not summed} \tag{1.6}$$

$$\mathbf{v} = \sum_{r=1}^{3} v^r \sqrt{g_{rr}} \frac{\mathbf{g}_r}{\sqrt{g_{rr}}} = \sum_{r=1}^{3} v_r \sqrt{g^{rr}} \frac{\mathbf{g}^r}{\sqrt{g^{rr}}} \qquad (1.7)$$

Then, since $\mathbf{g}_r/\sqrt{g_{rr}}$ and $\mathbf{g}^r/\sqrt{g^{rr}}$ are unit vectors, all components $v_r\sqrt{g^{rr}}$ and $v^r\sqrt{g_{rr}}$ (r not summed) will have the same physical dimensions. The physical components of a vector include the square root of a metric.

Euclidean Metric Tensor: $g_{km}(\theta_1, \theta_2, \theta_3)$ is a Measure of length in a reference system.

$$g_{km}(\theta_1, \theta_2, \theta_3) = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_m}. \qquad (1.8)$$

Given a line element dx^1, dx^2, dx^3. The length of the element ds is determined by Pythagoras' rule: $ds^2 = dx^i dx^i = \delta_{ij} dx^i dx^j$ and is independent of the coordinate system. If dx is transformed to another coordinate system where x_i is a function of new basis vectors $(\theta_1, \theta_2, \theta_3)$. The change in length dx is related to the transformed coordinate system as follows: $dx_i = \frac{\partial x_i}{\partial \theta_k} d\theta^k$.

Substitute this into *Pythagoras' Theorem.*

$$ds^2 = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \theta_k} \frac{\partial x_i}{\partial \theta_m} d\theta^k d\theta^m = g_{km} d\theta^k d\theta^m \qquad (1.9)$$

Notice, the *Euclidean Metric* is symmetric $g_{km} = g_{mk}$.

Principal Values: The principal values of a tensor lie on the principal planes. They are the maximum and minimum values of the system. Let $\mathbf{v}$ define a principal axis and let σ be the corresponding principal value. Then the vector acting on the surface normal to $\mathbf{v}$ has components $\tau_{ij} v_j$ and the components of only the principal components σv_i.

$$(\tau_{ij} - \sigma \delta_{ji}) v_j = 0 \qquad (1.10)$$

Since τ_{ij} as a matrix is real and symmetric, therefore there exist three real-valued principal stresses and a set of orthonormal principal axes.

A solution to $\mathbf{v}$ has a nonzero solution if and only if

$$|\tau_{ij} - \sigma \delta_{ij}| = -\sigma^3 + \mathbf{I}_1 \sigma^2 - \mathbf{I}_2 \sigma + \mathbf{I}_3 = 0 \qquad (1.11)$$

Tensor Invariants: $\mathbf{I}$ are the invariants of a tensor.

$$\mathbf{I}_1 = \tau_{11} + \tau_{22} + \tau_{33}$$

$$\mathbf{I}_2 = \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{11} \end{vmatrix}$$

$$\mathbf{I}_1 \quad = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} \tag{1.12}$$

The invariants can also be written in terms of principal values.

Strain Energy: The strain energy W is a scalar function of all the variables which cause strain in a material. For a linear elastic material it is a linear function of the strains: $\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij}$.

1.1.3   Tensors

Scalars, Contravariant Vectors, Covariant Vectors

In nonrelativistic physics there are quantities like mass and length which are independent of reference coordinates, these quantities are tensors.

Scalars, covariant vector fields, and contravariant vector fields are all examples of tensors.

Transformations of Tensors

Contravariant Tensor Field of Rank Two, t_{ij}:

$$\bar{t}_{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t_{mn}(\theta^1, \theta^2, \theta^3) \frac{\partial \theta^m}{\partial \bar{\theta}^i} \frac{\partial \theta^n}{\partial \bar{\theta}^j} \tag{1.13}$$

Contravariant Tensor Field of Rank two, t^{ij}:

$$\bar{t}^{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t^{mn}(\theta^1, \theta^2, \theta^3) \frac{\partial \bar{\theta}^i}{\partial \theta^m} \frac{\partial \bar{\theta}^j}{\partial \theta^n} \tag{1.14}$$

Mixed tensor field of Rank Two, t^i_j:

$$\bar{t}^i_j(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t^m_n(\theta^1, \theta^2, \theta^3) \frac{\partial \bar{\theta}^i}{\partial \theta^m} \frac{\partial \theta^n}{\partial \bar{\theta}^j} \tag{1.15}$$

Transformations of the kronecker delta and permutation symbol tensors in general coordinates:

$$g_{ij} = \frac{\partial x^m}{\partial \theta^i} \frac{\partial x^n}{\partial \theta^j} \delta_{mn} = \frac{\partial x^n}{\partial \theta^i} \frac{\partial x^n}{\partial \theta^j} \tag{1.16}$$

$$g^{ij} = \frac{\partial x^i}{\partial \theta^m} \frac{\partial x^j}{\partial \theta^n} \delta^{mn} = \frac{\partial x^i}{\partial \theta^n} \frac{\partial x^j}{\partial \theta^n} \tag{1.17}$$

$$g^i_j = \frac{\partial x^i}{\partial \theta^m} \frac{\partial x^n}{\partial \theta^j} \delta^m_n = \delta^i_j \tag{1.18}$$

$$\epsilon_{ijk} = \frac{\partial x^r}{\partial \theta^i} \frac{\partial x^s}{\partial \theta^j} \frac{\partial x^t}{\partial \theta^k} e_{rst} = e_{ijk} \left| \frac{\partial x^m}{\partial \theta^n} \right| = e_{ijk} \sqrt{g} \tag{1.19}$$

$$\epsilon^{ijk} = \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} \frac{\partial \theta^k}{\partial x^t} e^{rst} = e^{ijk} \left| \frac{\partial \theta^m}{\partial x^n} \right| = \frac{e^{ijk}}{\sqrt{g}} \tag{1.20}$$

where g is the value of the determinant $|g_{ij}|$ and is positive for any proper coordinate system $g = |g_{ij}| > 0$.

Tensor fields of higher ranks: A field with contravariant rank p and covariant rank q and the rank of the tensor is $r = p + q$. The components in any two coordinate systems is related by.

$$\bar{t}^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \frac{\partial \bar{\theta}^{\alpha_1}}{\partial \theta^{k_1}} \cdots \frac{\partial \bar{\theta}^{\alpha_p}}{\partial \theta^{k_p}} \cdot \frac{\partial \theta^{m_1}}{\partial \bar{\theta}^{\beta_1}} \cdots \frac{\partial \theta^{m_q}}{\partial \bar{\theta}^{\beta_q}} t^{k_1 \ldots k_p}_{m_1 \ldots m_q} \tag{1.21}$$

Tensor Theorem: Let $A^{\beta_1 \ldots \beta_s}_{\alpha_1 \ldots \alpha_r}$, $B^{\beta_1 \ldots \beta_s}_{\alpha_1 \ldots \alpha_r}$ be tensors. The equation

$$A^{\beta_1 \ldots \beta_s}_{\alpha_1 \ldots \alpha_r}(\theta^1, \theta^2, \ldots, \theta^n) = B^{\beta_1 \ldots \beta_s}_{\alpha_1 \ldots \alpha_r}(\theta^1, \theta^2, \ldots, \theta^n) \tag{1.22}$$

is a tensor equation. Thus if it is true in some coordinate system then it is true in all coordinate systems which are in one-to-one correspondence with each other.

Tensor Products

The product of tensors is a tensor since it transforms like a tensor.

Dot Product: The dot product reduces the order of the tensors involved: $\vec{a} \cdot \vec{b} = a^i \vec{g}_i \cdot b_j \vec{g}^j = a^i b_j (\vec{g}_i \cdot \vec{g}^j) = a^i b_i$ or $a^i \vec{g}_i \cdot b^i \vec{g}_j = a^i b^i g_{ij}$.

Cross Product: The cross product preserves the order of the tensors involved: $\vec{a} \times \vec{b} = a^i b_j (g_i \times b^j)$. For cartesian coordinates: $\vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \vec{e}_k$. See the transformation of permutation symbol in the glossary.

Dyadic: The dyadic increases the order of the tensors involved. $a_i b_j = T_{ij}$

Covariant Differentiation of Vector Fields

Differentiation in generalized coordinates caries an extra term. Without this extra term the derivative is not a tensor.

$$\xi^i|_j = \frac{\partial \xi^i}{\partial x^j} + \Gamma(i, j, \alpha)\xi^\alpha \tag{1.23}$$

Euclidean Christoffel Symbols:

$$\Gamma^i_{\alpha\beta}(x^1, x^2, x^3) = \frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right) \tag{1.24}$$

The Euclidean Christoffel Symbol does not transform as a tensor. It transforms as follows:

$$\bar\Gamma^i_{\alpha\beta}(\bar x^1, \bar x^2, \bar x^3) = \bar\Gamma^\lambda_{\mu\nu}(x^1, x^2, x^3)\frac{\partial x^\mu}{\partial \bar x^\alpha}\frac{\partial x^\nu}{\partial \bar x^\beta}\frac{\partial \bar x^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar x^\alpha \partial \bar x^\beta}\frac{\partial \bar x^i}{\partial x^\lambda} \tag{1.25}$$

Thus,the christoffel term is required for the derivative in general coordinates to transform as a tensor:

$$\frac{\partial \bar\xi^i}{\partial \bar x^a} + \bar\Gamma^i_{m\alpha}\bar\xi^m = \left(\frac{\partial \xi^\lambda}{\partial x^\mu} + \Gamma^\lambda_{s\mu}\xi^s\right)\frac{\partial x^\mu}{\partial \bar x^\alpha}\frac{\partial \bar x^i}{\partial x^\lambda} \tag{1.26}$$

Thus, $\xi^i|_\alpha$ is a covariant derivative of a contravariant tensor. The covariant derivative of a covariant vector field is written as follows.

$$\xi_i|_a = \frac{\partial \xi_i}{\partial x^\alpha} - \Gamma^\sigma_{i\alpha}\xi_\sigma \tag{1.27}$$

More generally the covariant derivative of a tensor $T^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}$ of rank $p + q$, contravariant of rank p, covariant of rank q can be written as follows

$$T^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}|_\gamma \qquad = \frac{\partial T^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}}{\partial x^\gamma} + \Gamma^{\alpha_1}_{\sigma\gamma}T^{\sigma\alpha_2...\alpha_p}_{\beta_1\beta_2...\beta_q} + \cdots$$

$$\Gamma^{\alpha_p}_{\sigma\gamma}T^{\alpha_1...\alpha_{p-1}\sigma}_{\beta_1...\beta_{q-1}\beta_q} - \Gamma^\sigma_{\beta_1\gamma}T^{\alpha_1\alpha_2...\alpha_p}_{\sigma\beta_2...\beta_q} - \cdots$$

$$-\Gamma^\sigma_{\beta_q\gamma}T^{\alpha_1...\alpha_{p-1}\alpha_p}_{\beta_1...\beta_{q-1}\sigma} \tag{1.28}$$

This derivative is contravariant of rank p, and covariant of rank $q+1$. Notice that the christoffel symbols are zero in the cartesian coordinates.

1.1.4   Kinematics

Lagrangian: description of motion is constructed by following individual particles *particle point of view* — Spatial description.

$$\vec r = \vec f^l(\vec x^0, t) \tag{1.29}$$

Eulerian: description of motion constructed by observing the passage of particles through a fixed position in space the *field point of view* — Material description.

$$\vec{r} = \vec{f}^e(\vec{x}, t) \tag{1.30}$$

There is a correspondence between the Lagrangian and Eulerian description. The particle which arrives in the Eulerian observed field at time t can be described by its initial condition $\vec{x}^0$ (or marker or color) using the Lagrangian description: $x_i = f_i^l(\vec{x}^0, t)$. The field position is now described in terms of the markers of the particle and the Eulerian description can be formulated in the Lagrangian perspective: $x_i = f_i^e(f_1^l(\vec{x}, t), f_2^l(\vec{x}, t), f_3^l(\vec{x}, t), t)$. The Lagrangian can be solved using an inverse procedure.

Material derivative: must take in account the particles moving in and out of the reference frame

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \tag{1.31}$$

1.1.5   Kinetics

The Governing Equations of Continuum Mechanics describe the behavior of a body which remains continuous under the action of external forces ([8]), and satisfies the following assumptions.

Meso Scale: The material which makes up the body can be described by its average properties on a scale larger than the micro scale but smaller than the macro scale ([21]). For example, average energy remains constant independent of average or other mean value.

Force at a Point: The limit with respect to the meso scale:

$$\lim_{\delta a \to 0} \frac{\delta P}{\delta A} = 0, \quad \begin{array}{l} \text{where } \delta \text{P is the net force} \\ \text{that acts on the area } \delta \text{A} \end{array} \tag{1.32}$$

Moment at a Point: In materials with high stress gradients the moment at a point cannot be considered. However, for most material the following limit in the meso scale is acceptable:

$$\lim_{\delta a \to 0} \frac{\delta M}{\delta A} = 0, \quad \begin{array}{l} \text{where } \delta \text{M is the net} \\ \text{moment that acts} \\ \text{on the area } \delta \text{A} \end{array} \tag{1.33}$$

Materials which do not have this behavior are considered in theorems related to the *Cosserat Medium*.

Points on the body can be described with respect to material or spatial reference systems. Material equations of state describe the body in terms of its initial configuration (*Lagrangian: z^k*). Spatial equations describe the body in terms of its final configuration (*Eulerian: x^a*).

In material form the conservation of mass is $\rho_0 = \rho \mathbf{J}$ so $\rho = \rho_0 \frac{|g_{ij}|^{1/2}}{|g_{bc}|^{1/2}} \left| \frac{\partial z^k}{\partial x^a} \right|$.

Conservation of Mass: Mass can neither be created or destroyed. Thus, the change of mass in a control volume (cv) is equal to the amount of mass convected into it through its surface (s) with normal $\vec{n}$.

$$\frac{\partial}{\partial t} \int_{cv} \rho \; \mathrm{dV} = - \int_{s} \rho \vec{v} \cdot \vec{n} \; \mathrm{dS} \tag{1.34}$$

Using *Green's Theorem* and considering that by definition the control volume does not change with time.

$$\frac{\partial \rho}{\partial t} = \; -\nabla \cdot \rho \vec{v} = \; -\vec{v} \cdot \nabla \rho \; + \; -\rho \nabla \cdot \vec{v} \tag{1.35}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot \vec{v}) = 0. \tag{1.36}$$

Conservation of Linear Momentum: Surface forces and body forces are the only sources of linear momentum. The change of linear momentum in a control volume (cv) is equal to the linear momentum convected into the volume over its surface (s) described by the normal $\vec{n}$, the traction tensor ($\widetilde{P}$) and the body forces ($\vec{b}$).

$$\begin{aligned} \frac{\partial}{\partial t} \int_{cv} \rho \vec{v} \; \mathrm{d}V = &- \int_{s}(\rho \vec{v}) \; \vec{v} \cdot \vec{n} \; \mathrm{d}S \\ &+ \int_{S} \widetilde{P} \cdot \vec{n} \; \mathrm{d}S + \int_{cv} \rho \vec{b} \; \mathrm{d}V \end{aligned} \tag{1.37}$$

Following the procedure used for conservation of mass,

$$\frac{\partial \rho \vec{v}}{\partial t} = -\nabla \cdot (\rho \vec{v}\vec{v}) + \nabla \cdot \widetilde{P} + \rho \vec{b} \tag{1.38}$$

Using conservation of mass:

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \nabla \cdot \widetilde{P} + \rho \vec{b}. \tag{1.39}$$

Conservation of Angular Momentum: Following the same procedure as for linear momentum and using previous laws, this conservation law requires that the stress tensor ($\widetilde{P}$) be symmetric when it describes a non-*Cosserat Medium*.

Conservation of Energy: Energy can neither be created or destroyed only changed in form. The change of kinetic energy and internal energy (e) in a control volume (cv) is equal to the work done on the surface by the traction ($\widetilde{P}$) and by the body forces ($\vec{b}$) and the heat convected into the volume and heat created by body($\vec{r}$) ([14]).

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t} \int_{cv} (\rho \frac{v^2}{2} + \rho e) \; \mathrm{d}V = &\int_{s}(\widetilde{P} \cdot \vec{n}) \cdot \vec{v} \; \mathrm{d}S \\ &+ \int_{cv} \rho \vec{b} \cdot \vec{v} \; \mathrm{d}V - \int_{s} \vec{q} \cdot \vec{n} \; \mathrm{d}S + \int_{vc} \rho \vec{r} \end{aligned} \tag{1.40}$$

Using previous equations and methods:

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \mathrm{tr}(\widetilde{\mathrm{P}}\nabla \vec{\mathrm{v}}) - \nabla \cdot \vec{\mathrm{q}} + \rho \vec{\mathrm{r}}. \tag{1.41}$$

First Law of Thermodynamics: Using conservation of energy: the system change in kinetic energy K and potential energy U is equal to the power P and energy U in the system.

$$\dot{K} + \dot{U} = \bar{P} + \bar{Q} \tag{1.42}$$

Second Law of Thermodynamics: Given in cartesian coordinates:

$$\rho T \dot{\bar{S}} + q_{i,i} - \frac{1}{T} q_{i,i} T_{,i} - \rho r \geq 0 \tag{1.43}$$

S is the entropy and T is the temperature of the system. A process is *reversible* if the right hand side is equal to zero.

Potentials An energy potential is a partial derivative of the total system potential energy with respect to a system variable. In general:

$$\tau_k = \frac{\partial U}{\partial \alpha_k} \tag{1.44}$$

Helmholz Free Energy: Another representation of the energy potential

$$A = U - TS \tag{1.45}$$

The second law of thermodynamics is thus defined as:

$$\widetilde{\sigma} : \widetilde{\epsilon} - \rho(\dot{A} + \dot{T}S) - \frac{1}{T} \vec{q} \nabla T \geq 0 \tag{1.46}$$

Energy of an Elastic Body: The potential energy in an elastic body depends only on the strain and the temperature $A(\widetilde{\epsilon}, T)$.

The first law of thermodynamics reduces to: $\widetilde{\sigma} - \rho \frac{\partial A}{\partial \widetilde{\epsilon}} = 0$.

The second law of thermodynamics reduces to: $\frac{1}{T} \vec{q} \nabla T \leq 0$.

Potential Functions: are used to simplify the description of stress, the potential of an elastic body. These functions satisfy Equilibrium and Compatibility conditions automatically. [1] Boundary conditions must still be considered.

1.1.6 Stress & Strain Definitions

Strain

Strain is defined as elongation per unit length. The *displacement vector*r: $\vec{u} = \vec{r} - \vec{r}o + \vec{d}$. $\vec{r}_0$ is the vector to the undeformed $\vec{r}$ is the vector to the deformed

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[1]Maxwell Moreial: $\sigma = \nabla \times \Phi \times \nabla$; Airy Stress function: $\sigma_{11} = \phi_{,22}, \sigma_{22} = \phi_{,11}$ such that $\nabla^4 \phi = 0$ is the only remaining governing equation; and the Torsion function: $\sigma = \nabla \cdot \psi$ see section on St. Venant Torsion. Plane stress and plane strain can also be represented by the airy stress function using the Beltrami-Michell Equation.

position. $\vec{d}$ is the distance between the origins of the undeformed and deformed configurations, the origins are fixed.

$$\mathrm{d}\vec{u} = \mathrm{d}\vec{r} - \mathrm{d}\vec{r}_0 \quad \text{and} \quad \mathrm{d}\vec{u} = \mathrm{d}\vec{r}_0 \cdot \nabla\vec{u} = \mathrm{d}\vec{r} \cdot \nabla\vec{u} \tag{1.47}$$

By definition: the finial configuration is defined in terms of *spatial* coordinates $\vec{r} = z^i\vec{g}_i$, while the original is defined in terms of *material* $\vec{r}_0 = x^a\vec{g}_a$. Thus the change in the displacement vector can be written in terms of its derivative in the *spatial* or *material* coordinate system.

$$u_{i|k}\mathrm{d}z^k = \mathrm{d}z^i - \mathrm{d}x^{a=i} \tag{1.48}$$

$$u_{a|b}\mathrm{d}x^b = \mathrm{d}z^{i=a} - \mathrm{d}x^a \tag{1.49}$$

The strain tensor is a defined quantity. It is described by the changes in length of the vectors $\vec{r}_0$ and $\vec{r}$. Set $\mathrm{d}l = |\mathrm{d}r|$ and $\mathrm{d}l_0 = |\mathrm{d}r_0|$. This definition is convenient since it has the propoerties of a metric.

$$\mathrm{d}l^2 - \mathrm{d}l_0^2 = 2e_{ab}\mathrm{d}x^a\mathrm{d}x^b = 2h_{kl}\mathrm{d}z^k\mathrm{d}z^l \tag{1.50}$$

Solving for the *Green Strain Tensor* and the *Cauchy Strain Tensor* respectively:

$$e_{ab}(x,t) = \frac{1}{2}\left[g_{kl}\ z^k|_a z^l|_b - g_{ab}\right] \qquad h_{kl}(z,t) = \frac{1}{2}\left[g_{kl} - g_{ab}\ x^a|_k x^b|_l\right]. \tag{1.51}$$

Notice that e_{ab} and h_{kl} are symmetric tensors. In terms of deformation:

$$e_{ab}(x,t) = \frac{1}{2}\left[u_{c|a}u^c|_b + u_{b|a} + u_{a|b}\right] \qquad h_{kl}(z,t) = \frac{1}{2}\left[u_{k|l} + u_{l|k} - u_{m|k}u^m|_l\right] \tag{1.52}$$

Assuming small displacement gradients $e_{ab} \approx h_{kl}$.

Stress

Stress is the force per unit area. *Traction* is the force at a point. The *stress tensor* when operated upon the unit normal to a surface reveals the traction on that surface: $\widetilde{\sigma} \cdot \vec{n} = \vec{t}_n$. Stress tensors are defined in terms of initial and final configurations and forces.

name	symbol	physical characteristic	symmetric
	$\widetilde{P}^{kl}$	final forces on final area	yes
Piolla Kirchhoff I	$\widetilde{S}^{ab}$	initial forces on initial area	no
Piolla Kirchhoff II	$\widetilde{t}^{ab}$	$S_{ab} = t_{al}x^b_{,l}$ & $t^{al} = S^{ab}z^l_{,b}$	yes

If the stress tensor satisfies static equilibrium in a non-cosserat medium, it is symmetric. Proof of the symmetry can be shown using the conservation of angular momentum.

Constitutive Equations

The relationship between stress and strain is empirically defined. Either it is assumed to be linear elastic, or the contributions to the strain energy of the system are assumed to be conservative and dependent only on first order of strain.

Elastic: Let the strain energy function be defined only in terms of strain and temperature $W(\epsilon_{ij}, t)$. Thus strain is defined as follows:

$$\sigma_{ij} = \frac{\mathrm{d}W}{\mathrm{d}\epsilon_{ij}} = \frac{\partial W}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial \epsilon_{ij}} \tag{1.53}$$

Generalized Hooke's Law: The simplest form of a linear relationship between stress and strain is:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{1.54}$$

There are symmetries in the constant. Due to the conservation of angular momentum $\sigma_{ij} = \sigma_{ji}$, similarly $c_{ijkl} = c_{jikl}$. By definition strain is symmetric $\epsilon_{kl} = \epsilon_{lk}$, similarly $c_{ijkl} = c_{ijlk}$. Using strain energy definition $c_{ijkl} = \frac{\partial^2 W}{\epsilon_{ij}\epsilon_{kl}} = \frac{\partial^2 W}{\epsilon_{kl}\epsilon_{ij}}$, thus $c_{ijkl} = c_{klij}$. There are 21 independent constants in the linear stress strain relationship.

Plane of Symmetry: If a plane of symmetry occurs in a chosen coordinate system the strain in $\{x_1, x_2, x_3\}$ is the same as that in $\{-x_1, x_2, x_3\}$. The coordinate transformation between these two coordinate systems:

$$\widetilde{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.55}$$

The transformation of the constant is as follows:

$$\bar{C}_{ijkl} = Q_{ai}Q_{bj}Q_{ck}Q_{dl}C_{abcd} \tag{1.56}$$

If there is a plane of symmetry $\bar{C}_{ijkl} = C_{ijkl}$. Thus: $C_{1112} = C_{1113} = C_{1222} = C_{1223} = C_{1233} = C_{1322} = C_{1323} = C_{1333} = 0$. The number of independent constants is reduced to 13.

Orthotropic: Using the transformations for the single degree of symmetry it can be shown that the 21 independent constants are reduced to 9.

Linear Isotropic: Let the strain energy function be defined only in terms of strain $W(\epsilon_{ij})$. In an isotropic system the energy does not depend on direction. Therefore the strain energy is a function of the invariants of the strain tensor $W(\mathbf{I}_\epsilon, \mathbf{II}_\epsilon, \mathbf{III}_\epsilon)$.

$$\sigma_{ij} = \frac{\partial W}{\partial \mathbf{I}_\epsilon}\frac{\mathrm{d}\mathbf{I}_\epsilon}{\mathrm{d}\epsilon_{ij}} + \frac{\partial W}{\partial \mathbf{II}_\epsilon}\frac{\mathrm{d}\mathbf{II}_\epsilon}{\mathrm{d}\epsilon_{ij}} + \frac{\partial W}{\partial \mathbf{III}_\epsilon}\frac{\mathrm{d}\mathbf{III}_\epsilon}{\mathrm{d}\epsilon_{ij}} \tag{1.57}$$

Recall one form of the invariant of a tensor: $\mathbf{I}_\epsilon = \epsilon_{kk}$; $\mathbf{II}_\epsilon = \frac{1}{2}\epsilon_{kl}\epsilon_{kl}$; $\mathbf{III}_\epsilon = \frac{1}{3}\epsilon_{kl}\epsilon_{lm}\epsilon_{mk}$. Take derivatives to construct:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{kk}}\delta_{ij} + \frac{\partial W}{\partial \frac{1}{2}\epsilon_{kl}\epsilon_{kl}}\epsilon_{ij} + \frac{\partial W}{\partial \frac{1}{3}\epsilon_{kl}\epsilon_{lm}\epsilon_{mk}}\epsilon_{ik}\epsilon_{kj} = f_1\delta_{ij} + f_2\epsilon_{ij} + f_3\epsilon_{ik}\epsilon_{kj} \tag{1.58}$$

For a linear material there can't be any nonlinear terms. Thus, $f_3 = 0$. Similarly f_2 must be a constant. Therefore the strain energy function must be in the form $W = C\frac{1}{2}\epsilon_{kl}\epsilon_{kl} + D$. Where C and D are constants. Finally $f_1 = \frac{\partial \frac{1}{2}\epsilon_{kl}\epsilon_{kl}}{\partial \epsilon_{nn}} = \epsilon_{mm}$.

$$\widetilde{\sigma} = \lambda \operatorname{tr}(\widetilde{\epsilon})\widetilde{I} + 2\mu\widetilde{\epsilon} \tag{1.59}$$

One Dimensional Analogy: The lamé constants can be found empirically for any material using experiments, one dimensional experiments are most conveneint. Then the constants E, G, K, ν are defined:

$$G = \frac{E}{2(1+\nu)} > 0 \quad \text{and} \quad K = \frac{E}{3(1-2\nu)} > 0 \tag{1.60}$$

Consequently poisson's ratio ν is defined in the range $-1 < \nu < 0.5$.

Compatibility:

In a domain $\nabla \times \widetilde{\epsilon} \times \nabla = 0$. In a multiply connected domain the line integrals about the openings must be zero. Given u_i calculating ϵ_{ij} results in a single value. Thus given ϵ_{ij}, u_i must be single valued. For two connected points on the domain an integral along the path connecting P_1 and P_2 must be path invariant.

$$\int_{P_1}^{P_2} \mathrm{d}\vec{u} = \vec{u}_{P_2} - \vec{u}_{P_1} = \int_{P_1}^{P_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} \tag{1.61}$$

Thus the integral around a closed path must be zero:

$$\int_{c_1} \mathrm{d}\vec{r} \cdot \nabla\vec{u} = \int_{c_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} \qquad \oint_{c_1+c_2} \mathrm{d}\vec{r} \cdot \nabla\vec{u} = 0 \tag{1.62}$$

Using Stoke's Theorem, for a singly connected domain

$$\oint \mathrm{d}\vec{r} \cdot \nabla\vec{u} = \int_S (\nabla \times \widetilde{\epsilon} \times \nabla)\mathrm{d}S = 0 \tag{1.63}$$

For a multiply connected domain, around each singularity:

$$\oint_{c_i} \mathrm{d}\vec{u} = 0 \tag{1.64}$$

The $\widetilde{\epsilon}$ is conserved in relation do displacements.

$$\widetilde{\epsilon} \quad = \quad \frac{1}{2}(\nabla\vec{u} + \vec{u}\nabla)$$

$$
\begin{aligned}
\nabla \times \widetilde{\epsilon} &= \frac{1}{2}(\nabla \times \nabla \vec{u} + \nabla \times \vec{u}\nabla) \\
\nabla \times \widetilde{\epsilon} \times \nabla &= \frac{1}{2}(\nabla \times \vec{u}\nabla \times \nabla) = 0 \qquad (1.65)
\end{aligned}
$$

Solvability Condition

According to Kirchhoff ([8], 160)

> If either the surface displacements or the surface traction are given, the solution for the problem of equilibrium of an elastic body...is unique in the sense that the state of stress (and strain) is determinate without ambiguity, provided that the magnitude of the stress (or strain) is so small that the strain energy function exists and remains positive definite.

The proof is by contradiction. Assume that two systems of displacements u_i' and u_i'' satisfy $\left(\frac{\partial W}{\partial e_{ij}}\right)_{,i} + \chi_i = 0$ — the equilibrium equation where χ_i is the effect of the body forces— and the boundary conditions: $S_u + S_\sigma$ defines the entire boundary surface; over S_u the values of u_i are given and over S_σ the traction $\vec{T}_i = \frac{\partial W}{\partial e_{ij}}v_j$ are specified. Then the difference $u_i' - u_i''$ satisfies the equation $\frac{\partial W}{\partial e_{ij}}_{,i} = 0$. Using Green's theorem and integrating over the volume $\int_V u_i \left(\frac{\partial W}{\partial e_{ij}}_{,i}\right) dV = 0$.

1.2 Inelasticity

1.2.1 References

- Viscoelasticity [7]

- Theory of Viscoelasticity: An Introduction [3]

- Plasticity Theory [16]

- Notes, Prof. Gautam Dasgupta, Columbia University, Fall 2000

1.2.2 Glossary

Elasticity: The strain energy in an elastic material depends only on the strain and the temperature.

Inelasticity: The strain energy of an inelastic material depends on the internal variables including but not limited to strain and temperature.

Heredity: A material whose deformation depends on the history of loading. According to the *axiom of non-reactivity* the deformation at the present time t is due only to the forces that acted in the past, not in the future.

Viscoelasticity: A viscoelastic material is exhibits time dependent behavior under the application of stress and strain.

> **Creep:** A creep function $c(t)$ describes the elongation over time produced by a sudden application of a constant force of unit magnitude at time $t = 0$.

> **Relaxation:** A relaxation function $k(t)$ describes the force required to produce an elongation which changes from zero to unity at time $t = 0$ and remains at unity thereafter.

Linear Heredity Material: A linear material whose deformation depends on the history of loading.

Relaxation Function: $k(t)$ the force required to produce an elongate which changes, at $t = 0$ from zero to unity and remaining at unity thereafter.

Creep Function: $c(t)$ the elongate produced by a sudden application at $t = 0$ of a constant force of unit magnitude.

Axiom of non-reactivity: The deformation at the present time t is due only to forces that acted in the past, not in the future. Thus,

$$c(t) = 0 \quad \& \quad k(t) = 0 \tag{1.66}$$

Bauschinger Effect: Elongation and compression in a specimen do not generate the same stresses. This is attributable to a form of residual stress occurring at grain boundaries ([12]:8).

Plasticity: Inelastic material which exhibits time independent unrecoverable deformations.

> **Yield Criteria:** The combination of primarily deviatoric stresses which causes yield in a material.

> **Flow Potential:** A scalar function of the deviatoric stresses and possibly the flow history. Its derivative in terms of stress times a positive scalar function gives the flow law.

> **Flow Law:** The rate of change of strain. It is a function of the internal variables including stress.

1.2.3   Inelastic constitutive relations ([16]: 55-68)

The strain at any point in a body is not completely determined by the current stress and temperature there as it is in an elastic solid. Thus strain is a function of stress, temperature and additional internal variables $\vec{\xi}$.
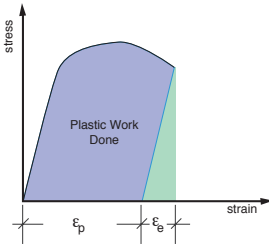
$$\widetilde{\epsilon} = \widetilde{\epsilon}(\widetilde{\sigma}, T, \vec{\xi}) \tag{1.67}$$

For a rate dependent body the internal variables also define the rate of evolution, or the equations of evolution

$$\dot{\xi}_\alpha = g_\alpha(\widetilde{\sigma}, T, \vec{\xi}).\tag{1.68}$$

Unlike for an elastic solid the relation $\widetilde{\epsilon} = \widetilde{\epsilon}(\widetilde{\sigma}, T, \vec{\xi})$ can not always be inverted to find the stress $\widetilde{\sigma} = \widetilde{\sigma}(\widetilde{\epsilon}, T, \vec{\xi})$. But, if it is possible then the equations of evolution, or rate equations can be described in terms of strain.

$$\dot{\xi}_\alpha = g_\alpha(\widetilde{\sigma}(\widetilde{\epsilon}, T, \vec{\xi}), T, \vec{\xi}) = \bar{g}_\alpha(\widetilde{\epsilon}, T, \vec{\xi}).\tag{1.69}$$

For inelastic bodies undergoing infinitesimal deformation it is possible to decompose elastic and inelastic strain:

$$\widetilde{\epsilon} = \widetilde{\epsilon}^e + \widetilde{\epsilon}^i\tag{1.70}$$

Newtonian viscosity used in fluid mechanics is formulated using this superposition: $\widetilde{\sigma} = K\mathrm{tr}(\widetilde{\epsilon})\widetilde{I} + 2\eta\widetilde{\epsilon}^v$. Where $\widetilde{\epsilon}^v$ is the viscous strain.

Flow rule from flow potential

It is possible in general to define a flow law or rate equation for $\widetilde{\epsilon}^i$ by assuming that $\epsilon^i = \epsilon^i(\xi)$ and applying the chain rule.

Generalized Potential and Generalized Normality ([16]:66)

Assume that the rate equations can be defined in terms of a potential Ω which depends only on thermodynamic forces $\vec{p}$

$$\dot{\vec{\xi}} = \frac{\partial \Omega}{\partial \vec{p}}\tag{1.71}$$

Using the Gibs function, or the complementary free energy -density $\chi = \rho^{-1}\widetilde{\sigma} : \widetilde{\epsilon} - \psi$, where ψ is the Helmholz free energy. The thermodynamic forces are

$$\vec{p} = \rho\frac{\partial \chi}{\partial \vec{\xi}} \quad \text{and} \quad \widetilde{\epsilon} = \rho\frac{\partial \chi}{\partial \widetilde{\sigma}}\tag{1.72}$$

Now using the cain rule find:

$$\dot{\widetilde{\epsilon}}^j = \frac{\partial \widetilde{\epsilon}}{\partial \vec{\xi}} \cdot \dot{\vec{\xi}} = \frac{\partial \vec{p}}{\partial \widetilde{\sigma}}\dot{\vec{\xi}}\tag{1.73}$$

Using the generalized potential find

$$\dot{\widetilde{\epsilon}}^j = \frac{\partial \Omega}{\partial \widetilde{\sigma}}\tag{1.74}$$

A sufficient condition fo the existence of a generalized potential was found by Rice in 1971. The condition is that each of the rate functions depend on the stress only through its own conjugate thermodynamic force $\vec{p}$. It is usually mathematically convenient to describe Ω as a convex function of $\vec{p}$. thus for any $\vec{p}^*$ such that $\Omega(\vec{p}^*) \leq \Omega(\vec{p})$, $(p - p^*) \cdot \dot{\vec{\xi}} \geq 0$.

Complete stress strain relations ([12])

An additional variable related to the strain rate is the yield surface. Given a yield criteria in the form

$$f(J_2', J_3') = c \tag{1.75}$$

where f does not depend on strain history.

$$\mathrm{d}f = \frac{\partial f}{\partial \widetilde{\sigma}} \mathrm{d}\widetilde{\sigma} = \frac{\partial f}{\partial J_2'} \mathrm{d}J_2' + \frac{\partial f}{\partial J_3'} \mathrm{d}J_3' \tag{1.76}$$

To assure that $\mathrm{d}\widetilde{\epsilon}^P$ is zero for a neutral change in stress assume

$$\mathrm{d}\widetilde{\epsilon}^p = \widetilde{G}\mathrm{d}f \tag{1.77}$$

where $\widetilde{G}$ is a symmetric tensor. $\mathrm{tr}(\widetilde{G}) = 0$ since hydrostatic stress does not produce plastic deformation. Thus $\widetilde{G}$ can be written in potential form

$$\widetilde{G} = h\frac{\partial g}{\partial \widetilde{\sigma}} \tag{1.78}$$

Where h and g are scalar functions of the deviatoric invariants and possibly of the strain-history.

$$\mathrm{d}\widetilde{\epsilon}^p = h\frac{\partial g}{\partial \widetilde{\sigma}}\mathrm{d}f. \tag{1.79}$$

Yield Criteria, Flow Rules and Hardening Rules ([16]:125-140)

It is convenient to write the stress in terms of a deviatoric stress.

$$\widetilde{S} = \widetilde{\sigma} - \frac{\mathrm{tr}(\widetilde{\sigma})}{\mathrm{tr}(\widetilde{I})}\widetilde{I} \tag{1.80}$$

Also for any yield function $f(\widetilde{\sigma}, \widetilde{\xi}) = \bar{f}(\widetilde{s}, \mathrm{tr}(\widetilde{\sigma}), \widetilde{\xi})$:

$$\frac{\partial f}{\partial \widetilde{\sigma}} = \frac{\partial \bar{f}}{\partial \widetilde{s}}\frac{\partial \widetilde{s}}{\partial \widetilde{\sigma}} + \frac{\partial \bar{f}}{\partial \mathrm{tr}(\widetilde{\sigma})}\frac{\partial \mathrm{tr}(\widetilde{\sigma})}{\partial \widetilde{\sigma}} = (\bar{\widetilde{f}} - \frac{1}{3}\mathrm{tr}(\bar{\widetilde{f}})\widetilde{I}) + \frac{\partial f}{\partial \mathrm{tr}(\widetilde{\sigma})}\widetilde{I} \tag{1.81}$$

where $\widetilde{f} = \frac{\partial f}{\partial s}$.

Initially Isotropic yield Criteria

The yield function f must depend only on the stress invariants of $\widetilde{\sigma}$, provided that the dependence is symmetric if there is no Bauschinger effect.

Yield Condition ([2])

Thus, a yield condition for a three dimensional solid can be formulated in terms of maximum shear. Using Mohr's circle or other tensor relations the maximum shear is given as $\frac{1}{2}|(\sigma_I - \sigma_{II})|$, $\frac{1}{2}|(\sigma_{II} - \sigma_{III})|$, or $\frac{1}{2}|(\sigma_{III} - \sigma_I)|$ where σ_I, σ_2 and σ_3 are the principle stresses. The actual yield condition depends on which of the given shears are negative. This is the Tresca Yield Condition.

Deformation energy may also be used to describe the yield condition. In the Mises-Hencky criterion

> Plastic flow will occur when the distortion-energy density in the material reaches the value corresponding to the yielding of a simple tensile specimen ([2],213).

The total strain energy density is $U = \frac{1}{2}\widetilde{\sigma} : \widetilde{\epsilon}$. For a linear elastic material $\widetilde{\sigma} = \lambda \text{tr}(\widetilde{\epsilon}) + 2\mu\widetilde{\epsilon}$. The deformation energy is related to the deviatoric component of stress and is $U^* = \frac{1}{2G}(\widetilde{\sigma} : \widetilde{\sigma} - \frac{1}{3}(\text{tr}\widetilde{\sigma})^2)$. For a uniaxial bar $U^* = \frac{(\sigma_0)^2}{6G}$. Now the Mieses-Hencky condition is formulated in terms of the principle components of stress:

$$\frac{1}{12G}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right] = \frac{(\sigma_0)^2}{6G}. \qquad (1.82)$$

Experiments have shown the Mises condition to be more closely correlated to experimental results.

In general the yield condition of a isotropic body is a function of the second J_2 and third J_3 invariants of the deviatoric components stress. The first invariant of stress is related to hydrostatic conditions. Thus in general:

$$f(J_2, J_3) = 0. \qquad (1.83)$$

To formulate a plastic stress-strain relation it is convenient to separate the elastic and plastic components of strain $\widetilde{\epsilon} = \widetilde{\epsilon}^e + \widetilde{\epsilon}^p$. Note the the plastic component is assumed to be unchanged during plastic deformation so $\text{tr}(\epsilon^p) = 0$.

Hardening Rules
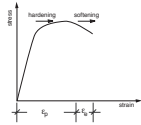
Hardening Rules: a specification of the dependence of the yield criterion on the internal variables, along with the rate equations of these variables.

Isotropic hardening [16]

Yield functions can be reduced to the form:

$$f(\widetilde{\sigma}, \xi) = F(\widetilde{\sigma}) - k(\vec{\xi}) \qquad (1.84)$$

Since only the yield stress is a function of the internal variables. The function $\widetilde{h}$ corresponds to the internal variables $\vec{\xi}$. So the plastic modulus H can be defined as

$$H = \begin{cases} k'(W_p)\widetilde{\sigma} : \widetilde{h} \\ k'(\bar{\epsilon}^p)\sqrt{\frac{2}{3}\widetilde{h} : \widetilde{h}} \end{cases} \tag{1.85}$$

1.2.4 Viscoplasticity ([16]:102–110)
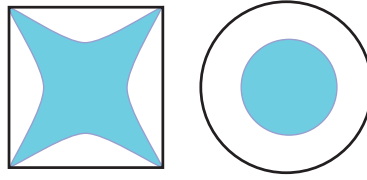
Yield surface

Figure 1.1: Yield Surfaces for Different Cross-Sections in Torsion

The yield surface separates regions where inelastic strain-rate tensor is zero and where it is not zero. In terms of yield function $f(\cdot)$, the yield surface is defined by $f(\widetilde{\sigma}, T, \vec{\xi}) = 0$.

Drucker's Postulate

If a unit volume of an elastic-plastic specimen under uniaxial stress is initially at stress $\widetilde{\sigma}$ and plastic strain at $\widetilde{\epsilon}^P$ and an "external agency" slowly applies an incremental load resulting in a stress increment $d\widetilde{\sigma}$ and subsequently removes it then $d\widetilde{\sigma} : d\widetilde{\epsilon} = d\sigma : (d\widetilde{\epsilon}^e + d\widetilde{\epsilon}^P)$ is the work performed by the external agency in the course of incremental loading and $d\widetilde{\sigma} : d\epsilon^P$ is the work performed in the course of the cycle consisting of the application and removal of the incremental stress.

A stable or work-hardened material is one in which the work done in an incremental loading is positive and the loading- unloading cycle is non-negative. The plastic strain increment dW_p is essentially positive since plastic distortion is an irreversible process in the thermodynamic sense. In general

$$d\widetilde{\sigma} : d\widetilde{\epsilon} > 0 \quad \text{and} \quad d\widetilde{\sigma} : d\widetilde{\epsilon}^P \geq 0 \quad \text{and} \quad W_p = \int \sigma_{ij} d\epsilon_{ij}^P. \tag{1.86}$$

For both work hardened and perfectly plastic material Drucker's Inequality holds

$$\dot{\sigma} : \dot{\epsilon}^P \geq 0 \tag{1.87}$$

The plastic strain rate can not act opposite to the stress rate. In general for a complete loading $\widetilde{\sigma}$ and unloading cycle $\widetilde{\sigma}^*$:

$$(\widetilde{\sigma} - \widetilde{\sigma}^*)\dot{\epsilon}^P \geq 0 \tag{1.88}$$

This is also know as the postulate of maximum plastic dissipation ([16],118).
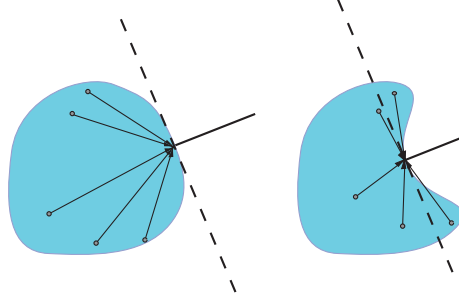

Figure 1.2: Drucker's Postulate and concavity

Consequences of the Maximum-Dissipation Postulate

If a yield surface is everywhere smooth, that is a well-defined tangent plane and normal direction exist at every point. For the inequality to be valid for all stresses inside the tangent line the rate of plastic strain must be directed along the outward normal associated with the tangent point. If there are any stresses lying on the outward side of the tangent the inequality is violated, that is the yield surface in stress space is convex.

1.2.5 Rate-independent plasticity ([16]: 112–140)

A perfectly plastic material is neither a function of time nor material imperfections. Thus it does not creep, relax, or exhibit Bauschinger's effect. The behavior of such a material depends on a yield function and its corresponding yield surface. Any change in the strain after yielding is a function only of hardening or softening of the material. The hardening is related to the plastic potential.

The Ideal Plastic Body

- Time Independent — No creep or relaxation

- No non-uniformity in the microscale and resulting differential hardening (Bauschinger Effect)

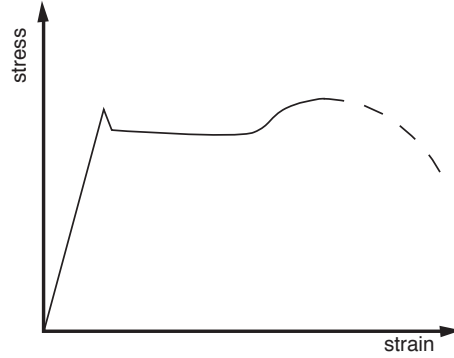- no side effects to plastic behavior

Figure 1.3: Possible Stress Strain relation for Steel

Yield Criteria: A law which defines the limit of elasticity under any possible combination of stress components. *Assume* for a plastic body yielding depends only on the deviatoric components of stress.

$$\widetilde{\sigma}' = \widetilde{\sigma} - \frac{\mathrm{tr}(\widetilde{\sigma})}{\mathrm{tr}(\widetilde{I})}\widetilde{I} \tag{1.89}$$

For an isotropic material which depends only on the invariants of the deviatoric stress tensor, the yield criteria depends only on $J_2' = \frac{1}{2}\widetilde{\sigma}' : \widetilde{\sigma}'$ and $J_3' = \frac{1}{2}\widetilde{\sigma}' : (\widetilde{\sigma}' \cdot \widetilde{\sigma}')$. To allow for symmetry the yield function must be an even function of J_3'

Introduction to Plasticity ([2]: 206–280)

The stress-strain relation under uniaxial loading condition can be found experimentally. For example the behavior of a thin specimen of steel may be similar to figure 1.3. Notice when yielding is reached σ_y the behavior changes, the relation between stress and strain does not follow $\sigma = E\epsilon$ as for the purely linear case. One representation constructed by matching experimental stress-strain curves is the *Ramberg-Osgood* Relation

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n. \tag{1.90}$$

For a perfectly plastic material $\sigma = \sigma_y$. Observations of experiments seem to show that plastic deformation is related only to shear stresses and not to hydrostatic pressure.

Plastic Potential

The plastic potential $g(\sigma_{ij})$ defines the ratios of components of plastic strain increments. If it is equal to $f(\sigma_{ij})$, the function which defines the yield locus.

In this condition the constant contours of the plastic potential define the yield locus.

$$d\epsilon_{ij}^P = h\frac{\partial f}{\partial \sigma_{ij}}df \tag{1.91}$$

Note that f must be independent of hydrostatic pressure if the plastic volume change is zero.

$$\frac{\partial f}{\partial \sigma_{ii}} = 0 \tag{1.92}$$

Also, if the function f is an even function — there is no Bauschinger effect, then the reversal of the sign of the stress reverses the sign of the stress increment. This the case for the Lévy-Mises or Reuss equations where $f = g = J_2' = \frac{1}{2}\sigma_{ij}'\sigma_{ij}'$ and $\frac{\partial f}{\partial \sigma_{ij}} = \sigma_{ij}'$.

Such an even, independent function f can be used inversely to find a unique plastic state of stress arising from a given plastic strain increment $d\epsilon_{ij}^P$. Let σ_{ij}^* be any other plastic state of stress

$$f(\sigma_{ij}^*) = f(\sigma_{ij}) = c \tag{1.93}$$

The work done by this $d\epsilon_{ij}^P$ in the strain $d\epsilon_{ij}^P$ is $dW_p^* = \sigma_{ij}^*d\epsilon_{ij}^P$. The stationary value for varying plastic states σ_{ij}^* is when

$$\frac{\partial}{\partial \sigma_{ij}^*}(\sigma_{ij}^*d\epsilon_{ij}^P - f(\sigma_{ij}^*)d\lambda) = 0 \tag{1.94}$$

where, using Lagrange's method, a constant multiplier $d\lambda$ has been added. Thus:

$$d\epsilon_{ij}^P = d\lambda\frac{\partial f(\sigma_{ij}^*)}{\partial \sigma_{ij}^*}. \tag{1.95}$$

And $d\lambda = hdf$.

1.2.6 Viscoelasticity

Models for creep and relaxation functions of Linear Viscoelastic Material [8]

Bolzmann solid viscoelastic materials retain linearity between load and deflection, but the linear relationship depends on a third parameter time. For this class of material the present state of deformation can not be determined completely unless the entire history of loading is known.

A one dimensional simple bar made of such a material behaves as follows when fixed at one end and subjected to a force in the direction of the axis at the the other end.

$$du(t) = c(t - \tau)\frac{dF}{dt}(\tau)d\tau \tag{1.96}$$

Here the force at time t is $F(t)$ and it is continuous and differentiable. In the small time period $d\tau$ the change in loading is $\frac{dF}{dt}d\tau$. Thus, the change in

elongation of the bar $du(t)$ is proportional to the time interval $(t - T)$ and the increase in force.

If the origin of time is the beginning of motion and loading the equations above can be integrated to derive the Ludwig Bolzmann (1844-1906) constitutive equations.
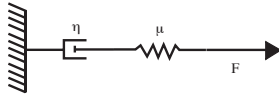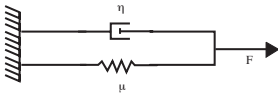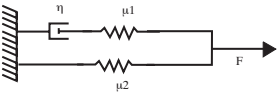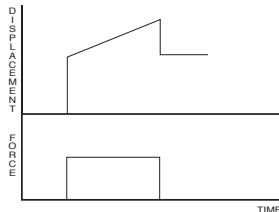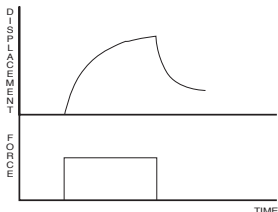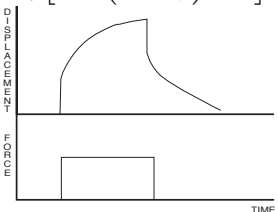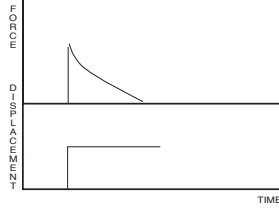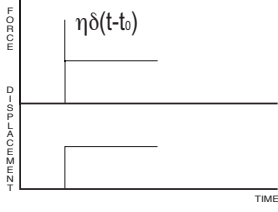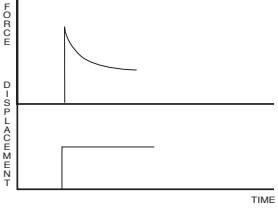
$$u(t) = \int_0^t c(t - \tau) \frac{dF}{dt}(\tau) d\tau \qquad (1.97)$$

similarly

$$F(t) = \int_0^t k(t - \tau) \frac{du}{dt}(\tau) d\tau \qquad (1.98)$$

Vito Voltera (1860-1940) extended this formulation —coining the term heredity law— for any functional relation of the type of the Bolzmann equation.

Viscoelasticity can be modeled with springs and dampers

	Maxwell model:	Voigt model:	Standard Linear model:
MODEL			
FORCE & DIS-PLACE-MENT	$\dot{u} = \frac{\dot{F}}{\mu} + \frac{F}{\eta}$	$F = \mu u + \eta \dot{u}$	$F + \tau_\eta \dot{F} = E_R(u + \tau_\sigma \dot{u})$
INITIAL CONDI-TION	$u(0) = \frac{F(0)}{\mu}$	$u(0) = 0$	$\tau_\eta F(0) = E_R \tau_\sigma u(0)$
CREEP	$c(t) = \left(\frac{1}{\mu} + \frac{1}{\eta}\right) = \mathbf{1}(t)$ 	$c(t) = \frac{1}{\mu}(1 - e^{-(\frac{\mu}{\eta})t})\mathbf{1}(t)$ 	$c(t) = \frac{1}{E_R}\left[1 - \left(1 - \frac{\tau_\eta}{\tau_\sigma}\right)e^{\frac{-t}{\tau_\sigma}}\right]\mathbf{1}(t)$
RELAX-ATION	$k(t) = \mu e^{-(\frac{\mu}{\eta})t}\mathbf{1}(t)$ 	$k(t) = \eta\delta(t) + \mu\mathbf{1}(t)$ $\eta\delta(t\text{-}t_0)$ 	$k(t) = E_R\left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\eta}\right)e^{\frac{-t}{\tau_\eta}}\right]\mathbf{1}(t)$

Where $\delta(t)$ is the Dirac-delta function, and $\mathbf{1}(t)$ is the unit step function and τ_η and τ_σ are constants.

Structural Problems of Viscoelastic Materials [8]

Generalized Hook load-displacement law using hereditary law.

$$u_i = \sum_{j=1}^{n} \int_0^t C_{ij}(t-\tau)\frac{d}{d\tau}F_j(\tau)d\tau, \qquad i = 1, \ldots, n. \tag{1.99}$$

$C_{ij}(t)$ is the creep function, the deflection $u_i(t)$ produced by a unit step function $F_j(t) = \mathbf{1}(t)$ acting at the point j. These creep functions are not always easy to deduce or measure.

Multiaxial behavior of viscoelastic material

The creep and relaxation functions can be formulated as fourth order tensors. If the material is isotropic then only two constants are needed to describe the behavior. Notice that internal variables are not required in this representation.

General Viscoelastic models ([3]: 9–14)

According to the hypothesis of fading memory the creep or relaxation effect decrease over time. Thus a the fourth-order relaxation function $\widetilde{\widetilde{G}}(t)$ or a fourth-order creep function $\widetilde{\widetilde{J}}(t)$ are continuously decreasing as a function of time.

$$\left|\frac{d\widetilde{\widetilde{G}}(t)}{dt}\right|_{t=t_1} \leq \left|\frac{d\widetilde{\widetilde{G}}(t)}{dt}\right|_{t=t_2} \quad \text{and} \quad \left|\frac{d\widetilde{\widetilde{J}}(t)}{dt}\right|_{t=t_1} \leq \left|\frac{d\widetilde{\widetilde{J}}(t)}{dt}\right|_{t=t_2} \quad \text{for } t_1 > t_2 > 0 \tag{1.100}$$

The difference between a *viscoelastic solid* and a *viscoelastic fluid* stated physically are: When a viscoelastic fluid is subjected to a fixed simple shear state of stress it responds with a steady state flow after the transient effects have died out, also when a fluid is subjected to a fixed simple shear state of deformation the shear stress state will eventually decay to zero: $\lim_{t\to\infty} G_1(t) = 0$. When a viscoelastic solid is subjected to a simple shear state of deformation, it will have a component of stress which remains nonzero as long as the state of deformation is maintained: $\lim_{t\to\infty} G_1(t) = C$ where C is a nonzero constant.

Subject a viscoelastic material to a simple shear state of deformation specified by the displacement components from the fixed reference configuration as:

$$u_1(\vec{x}, t) = \hat{u}X_2 h(t) \qquad u_2 = u_3 = 0 \tag{1.101}$$

where $h(t)$ is the unit step function. Using the infinitesimal strain displacement relations

$$\widetilde{\epsilon} = \frac{1}{2}\left(\nabla_{\vec{X}}\vec{u} + \vec{u}\nabla_{\vec{X}}\right). \tag{1.102}$$

The only nonzero relation between the components of stress and strain in a simple shear in cartesian coordinates is

$$s_{12}(t) = \left[\frac{G_1(t)}{2}\right]\hat{u} \tag{1.103}$$

where $G_{ijkl}(t) = \frac{1}{3}[G_2(t)-G_1(t)]\delta_{ij}\delta_{kl}+\frac{1}{2}[G_1(t)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]$. In an isotropic material the fourth order tensor is defined by two independent constants. Deviatoric stress in general is $\widetilde{s} = \int_{-\infty}^{t} G_1(t-\tau)\frac{d\widetilde{e}}{d\tau}d\tau$, where $\widetilde{e}$ is the deviatoric component of strain and strain $\widetilde{\sigma} = \int_{-\infty}^{t} G_2(t-\tau)\frac{d\widetilde{\epsilon}}{d\tau}d\tau$. Similar relations are formed using the creep functions. The relaxation function $G_1(t) = 0$ for $t < 0$.

1.2.7   Waves and Harmonic Response in Inelastic Material

One-dimensional plastic waves ([16]: 409–420)

If the propagation of longitudinal stress strain waves can be represented by a one-dimensional model. The inertia terms can be ignored, since the wave propagation problem is analogous to a creeping fluid problem where using scaling and Reynolds number it can be shown that the inertia term is relatively small compared with the viscous effects or shear stress. This assumption is not good near the ends of the bar.

Given x a Lagrangian coordinate along the $x-$axis, the corresponding small displacement strain ϵ and velocity v are:

$$\epsilon = \frac{\partial u}{\partial x} \quad \text{and} \quad v = \frac{\partial u}{\partial t} \tag{1.104}$$

According to the kinematic compatibility relation:

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial v}{\partial x}. \tag{1.105}$$

The equations of motion in the absence of body force reduce to:

$$\frac{\partial \sigma}{\partial x} = \rho\frac{\partial v}{\partial t} \tag{1.106}$$

Where σ is the nominal uniaxial stress and ρ is the mass density in the undeformed state. This reasoning can also be applied to the torsional problem.
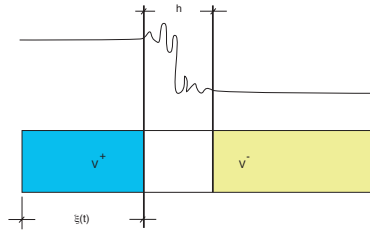
If the front is moving at a finite speed c in the positive $x-$ direction $c = \alpha'(t) > 0$ then the jump in the velocity v can be defined as in terms of the jump, or approximated with respect to a shock thickness h and the partial derivatives of the velocity v:

$$[v] = v^- - v^+ \quad [v] \approx \pm h\frac{\partial v}{\partial x} \quad [v] \approx \pm\frac{h}{c}\frac{\partial v}{\partial t}. \tag{1.107}$$

Stress and strain jumps at the shock front can be defined in terms of these relations:

$$[\epsilon] = \pm \frac{1}{c}[v], \quad [\sigma] = \pm \rho c[v] \quad \text{and} \quad \rho c^2 = \frac{[\sigma]}{[\epsilon]} \tag{1.108}$$

Notice that the final ratio depends only on the density and the wave speed not the velocity or the shock width.

Shock Front: occurs at a point $x = \alpha(t)$ on a bar the velocity v is discontinuous.

Shock-speed equation: The ratio between the jump in stress and the jump in strain.

$$\rho c^2 = \frac{[\sigma]}{[\epsilon]} \tag{1.109}$$

Harmonic response ([4])

In steady-state harmonic oscillations every forcing and response function is of the form

$$g(\vec{x}, t) = \bar{g}(\vec{x}, w)\mathrm{e}^{iwt}. \tag{1.110}$$

In general $\bar{g}$ can be a complex function. Correspondingly, for an isotropic viscoelastic body the Lamé constants are also frequency dependent $\lambda(w)$ and $\mu(w)$. If the viscoelastic characteristics of material can be assumed to be identical in bulk and shear then, the ratio of the frequency dependent constants is not a function of frequency, The bulk modulus is not a function of frequency:

$$\frac{\lambda(w)}{\mu(w)} = K = \frac{2\nu}{(1 - 2\nu)}. \tag{1.111}$$

For a Voigt Solid the energy loss per cycle of harmonic vibration is proportional to the excitation frequency, in a constant hysteretic solid the energy loss is independent of the frequency. The damping in a hysteretic solid can then be uncoupled in a multi degree of freedom system, as for structural or rate independent linear damping. The dynamic stiffness influence coefficients for Voigt and constant hysteretic solids can be found using the frequency dependent constitutive descriptions.

Viscoelastic correspondence principle ([3]: 187–221)

If a frequency response function of an elastic system is known, then the corresponding viscoelastic frequency response may be obtained from it directly.

Wave propagation in viscoelastic material on semi-infinite and infinite domains can be solved by integral transform, finite domain problems are far more complicated. Applying the findings from the infinite domains to large finite domains is usefully for practical problems.

Isothermal Wave Propagation

As for one-dimensional plastic waves, the stress-strain behavior about the strain discontinuity or shock front is used to solve the wave problem. As for the one-dimensional problem, the jump properties can be defined in terms of a jump width h, and the conservation of linear momentum can be applied across the jump.

The uniaxial viscoelastic constitutive relation is

$$\sigma(x,t) = \int_{-\infty}^{t} E(t-\tau)\frac{\partial\epsilon(x,\tau)}{\partial\tau}\mathrm{d}\tau. \tag{1.112}$$

Integrate by parts and substitute into the conservation of linear momentum

$$\left[E(0)\frac{\partial u(x,t)}{\partial x} + \int_{-\infty}^{t}\frac{\partial}{\partial t}E(t-\tau)\frac{\partial u(x,\tau)}{\partial x}\mathrm{d}\tau\right] = -\rho v\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{1.113}$$

Applying the strain displacement relation and assuming $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ is bounded and continuous on $0 \leq t \leq \infty$ then the integral term contributes nothing to the jump and, as for plastic waves:

$$E(0)\left[\frac{\partial u(x,t)}{\partial x}\right] = -\rho v\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{1.114}$$

Using kinematic compatibility as before the speed of propagation is given by $v = [E(0)/\rho]^{\frac{1}{2}}$ as in equation 1.108.

The result can also be found using the Laplace Transform.

$$\frac{\partial\sigma(x,t)}{\partial x} = \rho\frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{using the transform} \quad sE(s)\frac{\partial\bar{\epsilon}(x,s)}{\partial x} = \rho s^2\bar{u}(x,s) \tag{1.115}$$

The rod is initially assumed to be at rest

$$\frac{\partial^2\bar{u}}{\partial x^2} - \frac{\rho s}{\bar{E}(s)}\bar{u} = 0 \tag{1.116}$$

Taking the transform of the viscoelastic strain displacement relation

$$\frac{\partial^2\bar{\sigma}}{\partial x^2} - \frac{\rho s}{\bar{E}(s)}\bar{\sigma} = 0 \tag{1.117}$$

The same equation results from the laplace transform of strain, noticing that the particle velocity $\bar{v} = s\bar{u}$. The general solution of the equations is given by

$$(\bar{\sigma}, \bar{\epsilon}, \bar{u}, \bar{v}) = A(s)\mathrm{e}^{\Omega(s)x} + B(s)\mathrm{e}^{-\Omega(s)x} \quad \text{where} \quad \Omega(s) = \left[\rho s/\bar{E}(s)\right]^{\frac{1}{2}} \tag{1.118}$$

Here $A(s)$ and $B(s)$ are determined by the boundary conditions.

Reflection of Harmonic Waves

The fourier transformed equation of motion for an isotropic solid is is:

$$\mu^*(iw)\nabla^2\bar{\mathbf{u}} + [\lambda^*(iw) + \mu^*(iw)]\nabla(\nabla \cdot \bar{\mathbf{u}}) = -\rho w^2\bar{\mathbf{u}}. \qquad (1.119)$$

These equations are uncouples since the inertial terms can be neglected. The resulting equations of motions are

$$\mu^*\nabla^2\bar{\mathbf{u}} = -\rho w^2\bar{\mathbf{u}} \quad \text{and} \quad (\lambda^* + 2\mu^*)\nabla^2\bar{\mathbf{u}} = -\rho w^2\bar{\mathbf{u}}. \qquad (1.120)$$

These equations govern the propagation of shear S and irrotational P waves respectively.

Sinusoidal Oscillations in a Viscoelastic Material [8]

Assume a body is forced to perform simple harmonic oscillations. To find the steady state response assume that the forcing function has been acting on the body for an indefinitely longtime and that all initial transient disturbances have died out. Substitute $t - \tau = \xi$ into equation (1.14).

$$u(t) = \int_0^\infty c(\xi)\frac{dF}{dt}(t - \xi)d\xi \qquad (1.121)$$

This equation is true for any Bolzmann material excited with any function $F(x)$. When the forcing function is a simple harmonic oscillation the following equation is true $F(t) = F_0 e^{iwt}$.

$$\begin{aligned} u(t) &= \int_0^\infty c(\xi)iwF_0 e^{iw(t-\tau)}d\xi \\ &= iwF_o e^{iwt}\int_0^\infty c(\xi)e^{-iwt}d\tau \end{aligned} \qquad (1.122)$$

Since c(t)=0 when t¡0, replace the lower limit of the integral by $-\infty$ and writhe the integral in the conventional form of the fourier transformation.

$$\bar{c} = \int_{-\infty}^\infty c(\tau)e^{-iw\tau}d\tau \qquad (1.123)$$

Assuming that the fourier integral exists,

$$u(t) = iwF_0\bar{c}(w)e^{iw\tau} = u_0 e^{iw\tau} \qquad (1.124)$$

Thus, the ratio u_0/F_0 is a complex number and may be written as

$$\frac{u_0}{F_0} = \frac{1}{\mathcal{M}} = iw\bar{c}(w) = |w\bar{c}(w)|e^{-i\delta}. \qquad (1.125)$$

Where $\mathcal{M}$ is called the complex modulus of viscoelastic material. The angle δ is the phase angle by which the strain lags the stress. the tangent of δ is used to measure the internal friction of a linear viscoelastic material.

$$\tan\delta = \frac{imaginary\ part\ of\ \mathcal{M}}{real\ part\ of\ \mathcal{M}} \qquad (1.126)$$

also,

$$\frac{F_0}{u_0} = \mathcal{M} = iw\bar{k}(w) \tag{1.127}$$

where $\bar{k}(w)$ is the Fourier transform of the relaxation function

$$\bar{k} = \int_{-\infty}^{\infty} k(\tau)e^{-iw\tau}d\tau. \tag{1.128}$$

the relation between $\bar{c}(w)$ and $\bar{k}(w)$ is

$$-w^2\bar{c}(w)\bar{k}(w) = 1. \tag{1.129}$$

1.3   Mechanics of Solids

1.3.1   References

- Elasticity [19]

- Variational Calculus [15]

- Mechanics of Solids, Prof. Rene B. Testa, CU E4113, Fall 1998

1.3.2   Glossary

Plane Strain: the depth components of strain are non-existent. $\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$. A good approximation for rigid surfaces.

Plane Stress: the depth components of stress are non-existent. $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$. A good approximation for soft surfaces.

Curvature: $\kappa(x) = -w_{,xx}$ where w is the displacement normal to the x axis.

Center of Gravity: body forces act through the center of gravity of an object.

$$\bar{y} = \frac{\int_A y\,\mathrm{d}A}{\int_A \mathrm{d}A} \tag{1.130}$$

Moment of Inertia:

$$I_{yy} = \int_A y^2\mathrm{d}A \quad \text{and} \quad I_{xy} = \int_A xy\,\mathrm{d}A \tag{1.131}$$

Rotational Moment of Inertia:

$$J = \int_A r^2\mathrm{d}A \tag{1.132}$$

1.3.3 Theory

Elasticity

The governing equations for mechanics of solids are derived from continuum mechanics. In mechanics of solids elasticity theory is applied to solid objects. Concepts of *stress*, *deformation* and *constitutive equations* are used directly. Also, *solvability conditions* are used to prescribe the boundary conditions.

These equations:

- Strain displacement equations

$$\widetilde{\epsilon} = \frac{1}{2}(\nabla \vec{u} + \vec{u}\nabla) \tag{1.133}$$

- Conservation of linear momentum - Equilibrium

$$\nabla \cdot \widetilde{\sigma} + \vec{f} = \rho \ddot{\vec{u}} \tag{1.134}$$

- Constitutive (Linear Elastic)

$$\widetilde{\sigma} = \lambda \text{tr}(\widetilde{\epsilon})\widetilde{I} + 2\mu\widetilde{\epsilon} \tag{1.135}$$

- Compatibility

$$\nabla \times \epsilon \times \nabla = 0 \tag{1.136}$$

Simplifications of these equations are used to solve problems involving solids

Variational Calculus

Consider a function with fixed endpoints $x(t_0) = x_0$ and $x(t_1) = x_1$ which is a piceswise smooth scalar function defined for all $t \in [t_0, t_1]$. There exists a scalar function of this function $x(\cdot)$, its derivative $\dot{x}(\cdot)$ and time t: $f[t, x, \dot{x}]$ which is also defined throughout the entire interval $t \in [t_0, t_1]$. It is convenient that this new function $f(\cdot)$ is continuous and contains as many partial derivatives as necessary. The **functional** $J(\cdot)$ is now the sum of $f(\cdot)$ over the range of t.

$$J(x(\cdot)) \hat{=} \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)] \mathrm{d}t \tag{1.137}$$

The **global absolute minimum** of $J(\cdot)$ occurs at x^* if and only if $J(x * (\cdot)) \leq J(x(\cdot))$ $\forall x \in \{\text{domain}\}$. The **local minimum** of the integral occurs at x^* if within the immediate neighborhood of x^* the values of $J(\cdot)$ are greater than those at x^*. At a local minimum — a global minimimum is also a local minimum – the rate of change of the functional is zero, the function is **stationary**.

Using various theorems of the calculus of variations presented in the book the Euler-Lagrange form can be derived

$$f_{,x} = f_{,\dot{x}t}\dot{x}^*(t) + f_{,\dot{x}\dot{x}}\ddot{x}^*(t) \quad \forall t \in \mathbf{I} \tag{1.138}$$

Where $\mathbf{I}$ is the domain on which all the derivatives are continuous.

For mechanics problems the inverse function is of primary importance. If the function $x(t) = g(t, \alpha, \beta)$ is written in terms of a two paramater function, find the integrands $f(\cdot)$ which make the function $J(\cdot)$ stable. Assume that there exist continuous functions $\phi(\cdot)$ and $\psi(\cdot)$ to eliminate the constants.

$$\begin{aligned} x(t) &= g[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \\ \dot{x}(t) &= g_t[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))] \end{aligned} \tag{1.139}$$

Then $\ddot{x}(t) = G[t, x(t).\dot{x}(t)] = g_{tt}[t, \phi(t, x(t), \dot{x}(t)), \psi(t, x(t), \dot{x}(t))]$. And

$$f_{,x} - f_{,\dot{x}t} - \dot{x}(t)f_{,\dot{x}x} = Gf_{,\dot{x}\dot{x}} \tag{1.140}$$

This solution must hold for every initial condition $\{x[t_0], \dot{x}[t_0]\}$.

Letting $\mathbf{M}(t, x, \dot{x}) \hat{=} f_{,\dot{x}\dot{x}}(t, x, \dot{x})$. And assuming that the derivative operation is linear $f_{,xr} = f_{,rx}$, then

$$\mathbf{M}_{,t} + \dot{x}\mathbf{M}_{,x} + G\mathbf{M}_{,\dot{x}} + G_{,\dot{x}}\mathbf{M} = 0 \tag{1.141}$$

The general solution to this pde is $\mathbf{M} = \frac{\boldsymbol{\Phi}}{\boldsymbol{\Theta}}$. Where $\boldsymbol{\Phi}$ is differentialbe nonzero but otherwise arbitrary and

$$\boldsymbol{\Theta} \hat{=} \exp\left\{ \int G_{,\dot{x}}\left[t, g(t, \alpha, \beta), g_t(t, \alpha, \beta)\right] \mathrm{d}t \right\} \tag{1.142}$$

Finally functions $f(\cdot)$ can be found by integrating

$$f(t, x, \dot{x}) = \int_0^{\dot{x}} \int_0^q \mathbf{M}(t, x, p) \mathrm{d}p \mathrm{d}q + \dot{x}\lambda(t, x) + \mu(t, x) \tag{1.143}$$

Where $\lambda(\cdot)$ and $\mu(\cdot)$ are otherwise arbitrary functions which satisfy the continuity conditions and are defined on the required domain.

1.3.4   Applications

One Dimensional

In a bar under uniaxial loading the the stress is $\sigma = \frac{F}{A}$. The loading is constant along the bar. The displacement is related to the stress by the constitutive relationship: $\sigma = \mathrm{E}\varepsilon$. By definition $\varepsilon = u_{,x}$.

Orthotropic Material

A one dimensional force applied to an orthotropic material exhibits the following. The forces on the matrix and on the fiber $A_m\sigma_m = P_m$ and $A_f\sigma_f = P_f$. Using this separation $P = A_mE_m\epsilon_m + A_fE_f\epsilon_f = \epsilon_{ave}A_mE_m + A + fE_f) = \frac{P}{AE_x}(A_mE_m + A_fE_f)$. The bulk modulus can be rewritten as $E_x = \frac{A_m}{A}E_m + \frac{A_f}{A}E_f$. This is the basis for the **law of mixtures**: $E_x = (1 - \nu)E_m + \nu E_f$.

Airy stress function

The *AIRY* stress function ϕ describes two dimensional linear elastic small deformation plane stress or plane strain problems.

$$
\begin{aligned}
\sigma_x &= \phi_{,yy} \\
\sigma_y &= \phi_{,xx} \\
\tau_{xy} &= -\phi_{,xy}
\end{aligned}
\qquad (1.144)
$$

From axially symmetric stress equilibrium: $\nabla^2(\sigma_x + \sigma_y) = 0$. For axially symmetric problems: $\nabla^2\nabla^2\phi = 0$. The solution to this partial differential equation must satisfy the stress boundary conditions.

Elementary Beam Theory

Elementary beam theory can also be formulated as a two dimensional problem. The following assumptions are made: (1) plane sections remain planar,(2) small deformation.

From axial equilibrium: $\sigma_{x,x} + \tau_{,y} = 0$.

$$
\int_A (\sigma_{x,x}y + \tau_{,y}y)\mathrm{d}A = 0; \quad \frac{\partial}{\partial x}\int_A y\,\sigma_x\mathrm{d}A = -\int_A y\,\tau_{,y}\mathrm{d}A \qquad (1.145)
$$

The shear stress is assumed to be constant over the width of the beam:

$$
M = \int_A y\,\sigma_x\mathrm{d}A; \quad -b\int_{y_{\min}}^{y_{\max}} y\,\tau_{,y}\mathrm{d}y = -b(y\,\tau)|_{y_{min}}^{y_{max}} + \int_A \tau dA = 0 + V \quad (1.146)
$$

$$
\boxed{V = M_{,x}}
$$

From vertical equilibrium: $\sigma_{y,y} + \tau_{,x} = 0$.
Using assumption that plane sections remain planar.

$$
\int_A \sigma_{y,y}\mathrm{d}y = -\int_A \tau_{,x}\mathrm{d}A = -\frac{\partial}{\partial x}\int_A \tau\,\mathrm{d}A \qquad (1.147)
$$

$$
\boxed{P = -V_{,x}}
$$

The deformation is assumed to be linear in y so the solution must be in the following form:

$$
u(x,y) = C_1(x)y + C_2(x) \qquad (1.148)
$$

By definition the strain in the axial direction is:

$$
\varepsilon_x = u, x = C_1'(x)y + C_2'(x) = \kappa(x)y \qquad (1.149)
$$

Where κ is the curvature or the bar at x, thus by definition $\kappa = -w''$ and $\varepsilon_x = -w''y$. In basic elastic beam theory the shear deformation is zero: $\gamma = u_{,y} + u_{,x} = 0$. In the uniaxial case from the constitutive equation: $\sigma_x = E\,\varepsilon_x = -Ew''y$. Recall relation to moment, and assuming the curvature is constant over the area.

$$
M = -\int_A Ey^2 w'' dA = -EIw'' \qquad (1.150)
$$

The shear displacement is inconsistent because of the assumptions. Using $\tau = G\gamma$ one would assume that $\tau = 0$ because $\gamma = 0$, but this violates local equilibrium in x.

$$\sigma_{x,x} + \tau_{,y} = \frac{M'y}{\mathrm{I}} + \tau_{,y} = 0; \quad \tau_{,y} = \frac{Vy}{\mathrm{I}}; \quad \tau = \frac{VQ}{b\,\mathrm{I}} \qquad (1.151)$$

Compatibility is also unsatisfied in simple beam theory.

Timoshenko Beam Theory

The plane sections are still assumed to remain planar, but it is allowed a rotation ψ. Also, the shear deformation at the neutral is defined by $\gamma_{NA} = w' - \psi$.

From shear equilibrium: $\gamma_{xy} = u_{,y} + w_{,x}$. Define the shear τ separately at the neutral axis to avoid the inconsistency of simple beam theory: $\tau_{xy} = G\gamma_{xy}$ and $\tau_{NA} = G\gamma_{NA}$. Using the mean value theorem, the shear can be written as follows.

$$V = \int_A \tau_{xy}\mathrm{d}A = \int_A G\gamma_{xy}\mathrm{d}A = \tau_{ave}A = G\gamma_{ave}A \qquad (1.152)$$

Define a constant k such that $\tau_{ave} = k\,\tau_{NA}$.

$$V = kGA(w' - \psi) \qquad (1.153)$$

Using the definition of rotation of a plane section: $\varepsilon_x = y\psi'$. Using the definition of the moment and the constitutive equation:

$$M = \int_A y\sigma_x\mathrm{d}A = \int_A \mathrm{E}y^2\psi'\mathrm{d}A = -\mathrm{EI}\psi' = -\mathrm{EI}(-\frac{V'}{kGA} + w'') \qquad (1.154)$$

$$w'' = \frac{-P}{kGA} - \frac{M}{\mathrm{EI}}; \quad \mathrm{EI}w'' = -\frac{\mathrm{EI}}{kGA}P - M; \quad EIw^{IV} + \frac{\mathrm{EI}}{kGA}P'' = P \quad (1.155)$$

The relationship between the applied force P and the vertical displacement w is thus represented by a PDE and not an ODE. Still the approximation is not entirely valid since it assumes a shear force on the surface of the object which does not satisfy the free surface boundary condition.

Torsion

For a prismatic bar whose longitudinal axis is the x_3–axis and whose cross-section is simply connected and defined by a closed curve C in the x_1x_2–plane. **Simple theory** assumes no warping is only valid for circular sections. Assumption (1) no warping: uniform twist per unit length. $\theta(x) = \frac{\phi(x)}{L} = \phi_{,x}$. Thus, $\gamma = r\phi$, $\tau = \frac{Tr}{J}$ and $\phi = \frac{T}{GJ}$.

$$\begin{array}{lll} \text{at } x_3 = 0: & u_1 = u_2 = 0, \ \sigma_{33} = 0 \\ \text{at } x_3 = L: & u_1 = -\theta Lx_2, u_2 = \theta Lx_1, \ \sigma_{33} = 0; \\ \text{on } C, 0 < x_3 < L: & T_i = \sigma_{ij}n_j = \sigma_{i\alpha}n_\alpha = 0, \ \text{where } n_1 = x_{2,s}, n_2 = -x_{1,s} \end{array}$$
$$(1.156)$$

St. Venant Torsion

Assume (1) the shape of the section does not change (2) the twist is uniform $\phi(x) = \theta x$. Solve from the boundary conditions. First consider the torque alone *Pure Torsion*, then restrain $u(x)$ and develop σ_x as *Restrained Warping Tension*.

Warping (ψ): $u_x(y, z) = -\psi xz$ and $u_Z(y, z) = \psi xy$. The applied torque is $T = M_x$.

The boundary conditions in the case of pure torsion result in the following.
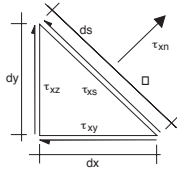
$$\tau_{xn} = \tau_{xy}\frac{dz}{ds} - \tau_{xz}\frac{dy}{ds} \tag{1.157}$$

$$\tau_{xs} = \tau_{xz}\frac{dz}{ds} + \tau_{xy}\frac{dy}{ds} \tag{1.158}$$

$$= (u_{x,y} - \theta_z)\frac{dz}{ds} + (u_{x,z} + \theta_y)\frac{dy}{ds} \tag{1.159}$$

Thermal Stresses

Thermal expansion or contraction causes strain without stress unless the boundary conditions inhibit the movement. A uniform temperature related strain is usually due to constraint of the ends. Linear stress related to temperature is usually related to curvatures, movements or external restrains. Non-linear stresses related to temperature are usually self equilibrating over the cross section.

For a restrained axial bar $\Delta L = \frac{P + P_t}{AE}L$. Restrained flexure $M'' - P$ and $w'' = -\frac{M + M_T}{EI}$. Stress can thus be written as $\sigma_x = \frac{M + M_T}{I}y + \frac{P + P_T}{A} - \alpha ET$. Where α is a coefficient of volume change. Stress occurs only if volume change is restrained. Using boundary conditions and the relationship $\pm V = \pm M' = \mp(EIw'' + M_T)'$. For a statically determinant beam with no applied forces $w'' = \frac{M_T}{EI}$ solving $M_T = \int_A \alpha EIy dA$. The Airy stress function related to restrained volumetric expansion is $\nabla^4 = -\alpha E \nabla^2 T$.

Pressure Vessels

An axisymmmetric pressure vessel endures only hydrostatic stresses. From equilibrium

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0 \tag{1.160}$$

Where σ_r is the radial stress and σ_t is the tangential or hoop stress. From geometric compatability the hoop strain and tangential strain are related

$$\epsilon_r = \frac{\left(U + \frac{du}{dr}dr\right) - u}{dr} = \frac{du}{dr} \quad \text{and} \quad \epsilon_t = \frac{2\pi(r + u) - 2\pi r}{2\pi r} = \frac{u}{r} \tag{1.161}$$

Apply the constitutive relation for linear elastic isotropic material and find:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{1.162}$$

Solve by applying the boundary conditions of either displacement or traction.

Chapter 2

Applied Mechanics

2.1 Fluid Mechanics

2.1.1 References

- Physical Fluid Dynamics [21]

- Mechanics of Fluids [18]

- Physics of Fluids, Prof. Adam Sobel, CU E4200, Fall 1999

- Mechanics of Fluids, Prof. Rene Chevray, CU E6100, Fall 2000

2.1.2 Glossary

Steady Motion: If at various points of flow all quantities (velocity, density ...) associated with the flow remain unchanged with time.

Unsteady Motion: not steady motion.

Streamline: at time instant (t_1) the line is parallel to the velocity field $\vec{v}(t_1)$.

Pathline: line described one particle as time passes (trajectory).

Streakline: a line which joins all particles at $t = t_1$ which passes through a given point at previous time $t \leq t_1$.

Timeline: a line which connects particles at time t_1 that passed through a line at time t_0.

The rate of deformation tensor: $\widetilde{D} = \nabla \vec{u}$. Divide the second order tensor into its symmetric and asymmetric parts.

$$\widetilde{D} = \frac{1}{2}(\widetilde{D} + \widetilde{D}^T) + \frac{1}{2}(\widetilde{D} - \widetilde{D}^T) \qquad (2.1)$$

37

The rate of strain tensor: is the symmetric part: $\widetilde{E} = \nabla\vec{u} + \vec{u}\nabla$.

Vorticity: is related to the asymmetric part: $(\nabla\vec{u} - \vec{u}\nabla)$, the vorticity can be written as a a vector: $\vec{\Omega} = \nabla \times \vec{u}$ also $\Omega_k = e_{ijk}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

Circulation: By mass flux analogy consider a surface inside a contour c. Derivation using Stokke's theorem.

$$\Gamma = \int_A \vec{\Omega} \cdot \mathrm{d}A = \int_A (\nabla \times \vec{v}) \cdot \mathrm{d}A = \oint_S \vec{v} \cdot \mathrm{d}S \qquad (2.2)$$

System: A system consists of a definite number of particles with a definite mass and it is distinguished from all other matter — its surroundings.

Control Volume: A control volume is a fixed region in space.

2.1.3   Summary of Equations of Motion

Conservation of Mass

The rate of inflow across the surface is equal to the rate of increase of the mass inside.

$$-\oint (\rho\vec{u}) \cdot \vec{n} dA = \frac{d}{dt} \int \rho dV \qquad (2.3)$$

$$\nabla \cdot \vec{u} = 0 \quad \text{for incompressible flow} \qquad (2.4)$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = 0 \quad \text{in general} \qquad (2.5)$$

Conservation of Momentum

The surface force and the body force is equal to the time rate of change in momentum energy and the net outflow of momentum across the surface.

$$-\oint \mathbf{p}\vec{n}dA + \oint \widetilde{S} \cdot \vec{n}dA + \int \vec{\mathbf{f}}\rho dV = \frac{d}{dt}\int \rho\vec{u}dV + \oint \vec{u}(\rho\vec{u} \cdot \vec{n})dA \qquad (2.6)$$

$\mathbf{p}$ is the pressure acting on the fluid particle. $\widetilde{S}$ is the stress not associated with pressure. And $\mathbf{f}$ is the body force.

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla\mathbf{p} + \vec{\mathbf{f}} \quad \text{for inviscid incompressible flow} \qquad (2.7)$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla\mathbf{p} + \frac{1}{\rho}\nabla \cdot \widetilde{S} + \vec{\mathbf{f}} \quad \text{in general} \qquad (2.8)$$

Conservation of Energy

for incompressible inviscid flow $\frac{D\frac{\vec{u}\cdot\vec{u}}{2}}{Dt}$ Material Change in Kinetic Energy. $\frac{1}{\rho}\nabla\cdot(\mathbf{p}\vec{u})$: Change in Potential Energy. $\vec{u}\cdot\mathbf{f}$: Work Done. $\mathbf{p}\frac{D}{Dt}\frac{1}{\rho}$: Dissipation Energy.

$$\frac{D}{Dt}(E + \frac{\vec{u}\cdot\vec{u}}{2}) + \frac{1}{\rho}\nabla\cdot(\mathbf{p}\vec{u}) - \vec{u}\cdot\mathbf{f} = Q' \quad \text{Energy} \tag{2.9}$$

$$Q' = \frac{DE}{Dt} + \mathbf{p}\frac{D}{Dt}\frac{1}{\rho} \quad \text{First law of thermodynamics} \tag{2.10}$$

Equations of State

$$E = C_t T \quad p = pRT \quad h = \frac{\mathbf{p}}{\rho} \quad \text{Perfect Gas} \tag{2.11}$$

Deformation and Rate of Strain

Experimental result show that fluid stress depends on the local rate of strain. The taylor expansion of the velocity:

$$U_i' = U_i + \frac{\partial U_i}{\partial x_j}\delta x_j + \frac{\partial^2 U_i}{\partial x_k \partial x_l}\frac{\delta x_k \delta x_l}{2!} + \cdots \tag{2.12}$$

Use only the first order term:

$$\vec{U}' - \vec{U} = \nabla\vec{U}\cdot\delta\vec{x} = \widetilde{D}\cdot\delta\vec{x} = \frac{1}{2}(\widetilde{E} + \widetilde{\Omega})\delta\vec{x} \tag{2.13}$$

Constitutive Equation

The Newtonian fluid is assumed linear and elastic in terms of the stress – strain relation. Recall $\widetilde{E} = \nabla\vec{u} + \vec{u}\nabla$,

$$\widetilde{S} = \mu\widetilde{E} + \frac{1}{2}\lambda\text{tr}(\widetilde{E})\widetilde{I} = \mu(\nabla\vec{u} + \vec{u}\nabla) + \lambda(\nabla\cdot\vec{u})\widetilde{I} \tag{2.14}$$

The volumetric stress can be written relative to the bulk modulus K.

$$\frac{\text{tr}(\widetilde{S})}{3} = (\lambda + \frac{2}{3}\mu)(\nabla\cdot\vec{u}) = K(\nabla\cdot\vec{u}). \tag{2.15}$$

Thus, the stress can be written in terms of the bulk modulus K and the velocities $\vec{u}$.

$$\widetilde{S} = -\frac{2}{3}\mu(\nabla\cdot\vec{u})\widetilde{I} + \mu(\nabla\vec{u} + \vec{u}\nabla) + K(\nabla\cdot\vec{u})\widetilde{I}. \tag{2.16}$$

Navier Stokke's Equation

Combining the constitutive equation and the conservation of linear momentum and assuming μ and K are constant:

$$
\begin{aligned}
\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} &= -\frac{1}{\rho}\nabla\mathbf{p} + \vec{\mathbf{f}} + \frac{1}{\rho}\nabla\cdot\left(-\frac{2}{3}\mu(\nabla\cdot\vec{u})\widetilde{I} + \mu(\nabla\vec{u}+\vec{u}\nabla) + K(\nabla\cdot\vec{u})\widetilde{I}\right)\\
&= -\frac{1}{\rho}\nabla\mathbf{p} + \vec{\mathbf{f}} + \nabla\cdot\left(-\frac{2}{3}\nu(\nabla\cdot\vec{u})\widetilde{I} + \nu\nabla\vec{u} + \nu\vec{u}\nabla + \frac{K}{\rho}(\nabla\cdot\vec{u})\widetilde{I}\right)\\
&= -\frac{1}{\rho}\nabla\mathbf{p} + \vec{\mathbf{f}} + \nu\nabla^2\vec{u} + \left(\frac{K}{\rho}+\frac{\nu}{3}\right)(\nabla(\nabla\cdot\vec{u})) \qquad (2.17)
\end{aligned}
$$

If the fluid flow is incompressible then the common form of the Navier Stokke's Equation is

$$
\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla\mathbf{p} + \vec{\mathbf{f}} + \nu\nabla^2\vec{u}. \qquad (2.18)
$$

Usually the bulk modulus K is very small compared to the viscous effects and is usually neglected.

Conservation of Angular Momentum

The incompressible Navier Stokke's equations can be rewritten if the body forces can be described by a potential Θ such that

$$
\begin{aligned}
\nabla\times\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} &= -\nabla\times\frac{1}{\rho}\nabla\mathbf{p} + \nabla\times\nabla\Theta + \nabla\times\nu\nabla^2\vec{u}.\\
\frac{\partial\vec{\Omega}}{\partial t} &= \vec{\Omega}\cdot\nabla\vec{u} + \nu\nabla^2\vec{\Omega} \qquad (2.19)
\end{aligned}
$$

Other theorems related to rotation include the **Helmholz Theorem** The circulation around a vortex tube must be the same at all cross sections. The kinematic result is valid for all viscid and inviscid flows. Vortex tubes and stream tubes do not coincide in general except for Beltrami Flows. Vortex tubes are similar to stream lines in that they are parallel to vorticity like stream lines are parallel to velocity.

The angular momentum along a loop is described by **Kelvin's Circulation Theorem**

$$
\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = -\oint\frac{\nabla P}{\rho} + \oint\Theta\cdot\mathrm{d}\vec{r} + \oint\nu\nabla^2\vec{v}\cdot\mathrm{d}\vec{r}. \qquad (2.20)
$$

Boundary Conditions

At the interface between two fluids the shear should be zero in the direction of the interface where $\widetilde{\tau} = \nu\nabla\vec{u}$. By a moving wall the speed of the fluid at the wall should equal that of the wall. At a free surface the change in the velocity profile is zero in the direction of the interface.

2.1.4 Types of Flow

Shear Driven Flow: Couette Flow

Solve Navier stokes equation for incompressible unidirectional flow to find that the velocity distribution is linear, 0 at the fixed wall and U_0 at the moving wall. In such a flow the pressure is hydrostatic.

Pressure Driven Flow: Poiseulle Flow:

Given a constant pressure gradient k, a viscosity ν, and a density ρ, the parabolic flow profile is proportional by $-\frac{k}{2\mu}$.

Flow in Concentric Circles

$$\Gamma(r) = \oint_c \vec{v}_\theta(r) \cdot R d\theta = 2\pi r \vec{v}_\theta(R) \quad \text{and} \quad V_\theta = \frac{A}{r} \tag{2.21}$$

$$\Gamma(r) = 2\pi R \frac{A}{R} = 2\pi A \quad \text{is constant} \tag{2.22}$$

For any contour not including the center the circulation is zero, no vorticity. For any simply connected contour including the center the circulation is constant.

Vortex lines move with the fluid. a vortex tube cannot end within the fluid it must end at a solid boundary or end on itself. On a given vortex the quantity $\frac{w}{\rho l}$ is constant, where w is the vorticity l is hte length of the line, and ρ is density.

Rigid body Rotation

$$\Omega = \Omega_0 \quad \text{is constant} \tag{2.23}$$

$$V_\theta = \frac{\Omega_0}{2} r \tag{2.24}$$

Potential Theory

Potential Flow Φ A method for solving flow calculations is in closed form. Assumptions required include that the applied forces are conservative, no dissipation. That is there are no viscous effects. Consequently the change in force potential is equal to the work done by conservative forces (the sign change is a matter of convention):

$$\Phi_a - \Phi_b = \int_a^b \vec{F} \cdot d\vec{s} \tag{2.25}$$

Thus, $\vec{F}$ as a the gradient of the potential $\vec{F} = -\nabla\theta$. In cartesian coordinates:

$$u = -\frac{\partial\theta}{\partial x} \quad \text{and} \quad v = -\frac{\partial\theta}{\partial y} \tag{2.26}$$

Irrotational flow

$$2w \hat{=} \nabla \times \vec{v} = 0 \tag{2.27}$$

In terms of potential flow $\nabla \times (-\nabla\phi)$ is automatically zero. Thus potential flow is irrotational. A boundary layer is not irrotational. The flow outside a rankine vortex is irrotational, inside the vortex is rotational.

Euler's Equation The navier stokke's equation for incompressible flow and only gravity body forces can be rewritten as:

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla\left(P + \gamma h\right) \tag{2.28}$$

In terms of potential Φ:

$$\frac{\partial(\nabla\Phi)}{\partial t} + (\nabla\Phi)\cdot\nabla(\nabla\Phi) = -\frac{1}{\rho}\nabla\left(P + \gamma h\right) \tag{2.29}$$

$$\nabla\left[\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\vec{v}\cdot\vec{v}) + \frac{P}{\rho} + gh\right] = 0 \tag{2.30}$$

For convenience set $q = \vec{u}\cdot\vec{u}$. Thus the flow is not a function of spatial coordinates.

$$\frac{q^2}{2} + \frac{P}{\rho} + gh - \frac{\partial\Phi}{\partial t} = F(t) \tag{2.31}$$

At a given time t the sum is constant everywhere, not just along streamlines as it is in Bernoulli's equation.

The Stream Function Ψ The flow rate at between a fixed point A and a variable point $P(x,y)$ is given by $P\hat{=}\Psi(x,y)$.

Streamline: along stream lines the stream function Ψ is constant, the flow rate is constant since streamlines are defined to be parallel to the flow velocity and using continuity.

By calculating the flow between two streamlines which are infinitely far apart the velocity can be defined in cartesian coordinates as:

$$v = \frac{\partial\Psi}{\partial x} \quad \text{and} \quad u = -\frac{\partial\Psi}{\partial y}. \tag{2.32}$$

And in polar coordinates as:

$$V_r = \frac{-1}{r}\frac{\partial\Psi}{\partial\theta} \quad \text{and} \quad V_\theta = \frac{\partial\Psi}{\partial r}. \tag{2.33}$$

Now notice that the potential and the stream function are related by the Cauchy-Riemann Equations:

$$\frac{\partial\theta}{\partial x} = \frac{\partial\Psi}{\partial y} \quad \text{and} \quad \frac{\partial\theta}{\partial y} = -\frac{\partial\Psi}{\partial x}. \tag{2.34}$$

Boundary Conditions Here $\vec{q}$ is the velocity of the fluid:

- For fixed boundaries, the velocity component normal to the boundary is zero at every point on the boundary. $\vec{q} \cdot \vec{n} = 0$ if $\vec{n}$ is the unit outward normal from the boundary, similarly $\frac{d\Phi}{dn} = 0$. There is no shear resistance at the boundary since the viscosity everywhere is by definition zero.

- For boundaries moving at velocity v the normal components of the velocities of the fluid and the wall must be equal.

$$\vec{q} \cdot \vec{n} = \vec{v} \cdot \vec{n} \quad \text{and} \quad (\vec{q} - \vec{v}) \cdot \vec{n} = 0. \tag{2.35}$$

- At the interface between two fluids velocity differences may exist since there is no viscosity but the pressure must be continuous across the interface.

Scaling

Creeping Flow: the inertial term is relatively small and can be neglected.

$$u^* = \frac{u}{U}, \quad \nabla^* = \frac{\nabla}{L}, \quad t^* = \frac{t}{L/U}, \quad \text{and} \quad \mathbf{p}^* = \frac{\mathbf{p}}{\frac{U^2}{L\mu}} \tag{2.36}$$

$$\frac{LU}{\nu} \frac{\mathrm{D}\vec{u}^*}{\mathrm{D}t^*} = \left(-\nabla^* \mathbf{p}^* + \vec{\mathbf{f}}^* + \nabla^{*2} \vec{u}^* \right). \tag{2.37}$$

2.2 Plates and Shells

Assumptions:

1. no deformation in the middle of the plate

2. points of the plate initially lying normal to the middle plane of the plate remain so

3. the normal stresses in the direction transverse to the plate can be disregarded

2.2.1 References

- Theory of Plates and Shells [20]

- Plates and Shells, Prof. Gautam Dasgupta, CU, Spring 1999

2.2.2 Glossary

Curvature: is defined as the inverse of the radius: $1/\rho$.

Characteristic Dimension: unique measures of the object. For example thickness, length or curvature.

Membrane Analysis: Under the application of uniform internal pressure the
stress distribution changes negligibly along the section — thus moments
can be neglected.

2.2.3   The Geometry of Shell Surfaces
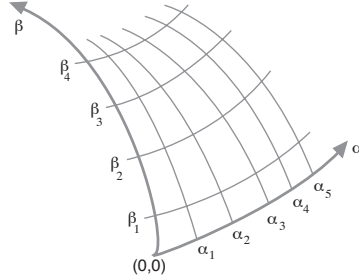
For simplicity choose an $\alpha - \beta$ coordinate sys-
tem which is **orthogonal** and traces along the
contour lines of the shell. In general a coordi-
nate grid is not necessarily most convenient in
orthogonal directions – such as in non-uniform
materials. The numbering of contour lines is
completely arbitrary. thus use the most conve-
nient not one based on lengths or equal spaces.
$\alpha - \beta$ orthogonal coordinate system
$\alpha_i \& \beta_i$ are constant along the contours

Defining a measure of a shell: **radius of curvature** and **thickness**.

Radius of curvature is defined by the changing quantity along a
coordinate line.

Thus, any point on the shell is defined by the vector $\vec{r}$ from the origin to its
position. Directional derivatives give the slope of the shell at a any point in any
given direction.
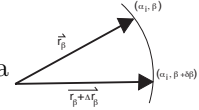
$$\vec{r},_a = \frac{\partial \vec{r}}{\partial \alpha} = \left| \frac{\partial \vec{r}}{\partial \alpha} \right| \vec{t}_a \quad \text{and} \quad \vec{r},_b = \frac{\partial \vec{r}}{\partial \beta} = \left| \frac{\partial \vec{r}}{\partial \beta} \right| \vec{t}_\beta \tag{2.38}$$

$\vec{t}_a$ and $\vec{t}_b$ are unit vectors in the given tangential directions the normal direction
is then $\vec{t}_n = \vec{t}_a \times \vec{t}_b$. They, like the radius of curvature are defined by the
value which is changing along the contour. The metric is the length element
$A = |r,_\alpha|$ and $B = |r,_\beta|$. Using this any length on the plate is defined as
$\mathrm{d}s^2 = A^2 \mathrm{d}\alpha^2 + B^2 \mathrm{d}\beta^2$.

The radius of curvature can be defined in terms of these
metrics and directional derivatives at any given point.
If $\mathrm{d}\psi_a$ small enough, the small angle approximation is
valid.
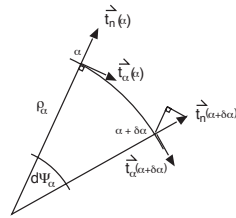
$$\mathrm{d}S_\alpha = R_\alpha \mathrm{d}\psi_\alpha = A \mathrm{d}\alpha \tag{2.39}$$

$$\lim_{\mathrm{d}\alpha \to 0} \frac{\vec{t}_n(\alpha + \mathrm{d}\alpha) - \vec{t}_n(\alpha)}{\mathrm{d}\alpha} = \left| \frac{\partial t_n}{\partial \alpha} \right| \vec{t}_\alpha = \vec{t}_{n},_\alpha \tag{2.40}$$

$$\mathrm{d}\psi_\alpha \vec{t}_\alpha = \vec{t}_n(\alpha + \mathrm{d}\alpha) - \vec{t}_n(\alpha) \tag{2.41}$$

$$\lim_{\Delta a \to 0} \frac{1}{A} \frac{\vec{t}_n(\alpha + \mathrm{d}\alpha) - \vec{t}_n(\alpha)}{\mathrm{d}\alpha} = \frac{1}{R_\alpha} \frac{\mathrm{d}\psi_\alpha \vec{t}_\alpha}{\mathrm{d}\psi_\alpha} = \frac{1}{A} \frac{\partial \vec{t}_n}{\partial \alpha} = \frac{1}{R_\alpha} \vec{t}_\alpha \tag{2.42}$$

$$\frac{1}{A}\vec{t}_{n,\alpha}\cdot\vec{t}_{\alpha} = \frac{1}{R_{\alpha}}\vec{t}_{\alpha}\cdot\vec{t}_{\alpha} = \frac{1}{R_{\alpha}} \quad \text{analogously} \quad \frac{1}{R_{\beta}}\vec{t}_{\beta}\cdot\vec{t}_{\beta} = \frac{1}{R_{\beta}} \qquad (2.43)$$

Principal Directions $\alpha^{*}-\beta^{*}$ are the directions of maximum and minimum curvature. They are orthogonal.

$$\frac{1}{R_{n}} = \frac{1}{2}\left(\frac{1}{R_{1}} + \frac{1}{R_{2}}\right) + \frac{1}{2}\left(\frac{1}{R_{2}} - \frac{1}{R_{1}}\right)\cos 2\alpha \qquad (2.44)$$

$$\frac{1}{R_{nt}} = \frac{1}{2}\left(\frac{1}{R_{1}} + \frac{1}{R_{2}}\right)\sin 2\alpha \qquad (2.45)$$

Gaussian curvature:

$$\frac{1}{R_{\alpha}}\cdot\frac{1}{R_{\beta}} = -\frac{\left(\frac{B_{,\alpha}}{A}\right)_{,\alpha} + \left(\frac{A_{,\beta}}{B}\right)_{,\beta}}{AB} \qquad (2.46)$$

Tools to Solve: Derivatives of Local Coordinate Vectors.

$$\begin{Bmatrix} \vec{t}_{\alpha,\alpha} \\ \vec{t}_{\alpha,\beta} \\ \vec{t}_{\beta,\alpha} \\ \vec{t}_{\beta,\beta} \\ \vec{t}_{n,\alpha} \\ \vec{t}_{n,\beta} \end{Bmatrix} = \begin{bmatrix} 0 & -\frac{A_{,\beta}}{B} & -\frac{A}{R_{a}} \\ 0 & \frac{B_{,\alpha}}{A} & 0 \\ \frac{A_{,\beta}}{B} & 0 & 0 \\ -\frac{B_{,\alpha}}{A} & 0 & -\frac{\beta}{R_{\beta}} \\ \frac{A}{R_{\alpha}} & 0 & 0 \\ 0 & \frac{B}{R_{\beta}} & 0 \end{bmatrix} \begin{Bmatrix} \vec{t}_{\alpha} \\ \vec{t}_{\beta} \\ \vec{t}_{n} \end{Bmatrix} \qquad (2.47)$$

Solve Procedure

$$\vec{t}_{\alpha,\alpha} = C_{11}\vec{t}_{\alpha} + C_{12}\vec{t}_{\beta} + C_{13}\vec{t}_{n} \quad \text{and} \quad \vec{t}_{\alpha,\alpha}\cdot\vec{t}_{\alpha} = C_{11} = 0 \qquad (2.48)$$

$$\text{and} \quad \vec{t}_{\alpha,\alpha}\cdot\vec{t}_{\beta} = C_{12} \quad \text{and} \quad \vec{t}_{\alpha,\alpha}\cdot\vec{t}_{n} = C_{13} \qquad (2.49)$$

Other values are from curvature.

2.2.4 Equilibrium

Separate a finite shell section. Apply **thin shell theory** assumptions: (1) Plane sections remain planar relative to the deformed midline of the shell (2) the shell is thin relative to the defining measurements of curvature and thickness. Consider the equilibrium over the section, notice that the **axial stresses** are linearly distributed and the shear is parabolic. Therefore **shear deformation** is 0 and there is no warping.

Membrane Forces

Bending Forces
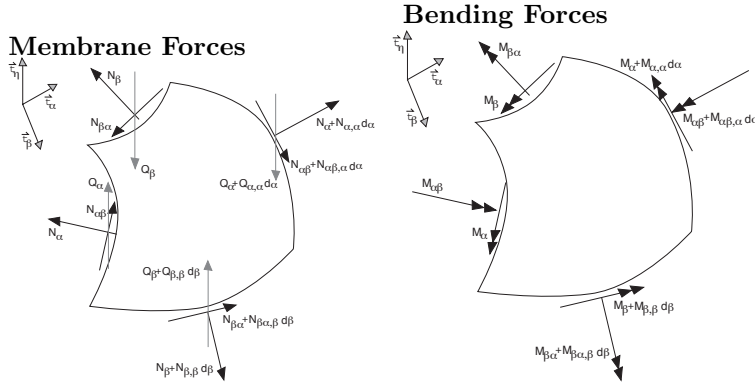
Equilibrium of Forces

$$
\begin{aligned}
\left(\left(\vec{t}_\alpha N_\alpha + \vec{t}_\beta N_{\alpha\beta} + \vec{t}_n Q_\alpha\right) B\mathrm{d}\beta\right)_{,\alpha}\mathrm{d}\alpha &\;+\; \\
\left(\left(\vec{t}_\beta N_\beta + \vec{t}_\beta N_{\beta\alpha} + \vec{t}_n Q_\beta\right) A\mathrm{d}\alpha\right)_{,\beta}\mathrm{d}\beta &\;+\; \\
\left(q_\alpha \vec{t}_\alpha + q_\beta \vec{t}_\beta + q_n \vec{t}_n\right) AB\mathrm{d}\alpha\mathrm{d}\beta &\;=\; 0
\end{aligned}
\tag{2.50}
$$

Equilibrium of Moments

$$
\begin{aligned}
\left(\left(M_\alpha \vec{t}_\beta + M_{\alpha\beta}\vec{t}_\alpha\right)B\mathrm{d}\beta\right)_{,\alpha}\mathrm{d}\alpha &\;+\; \\
\left(\left(M_\beta \vec{t}_\alpha + M_{\beta\alpha}\vec{t}_\beta\right)A\mathrm{d}\alpha\right)_{,\beta}\mathrm{d}\beta &\;+\; \\
Q_\alpha B\mathrm{d}\beta A\mathrm{d}\alpha + Q_\beta A\mathrm{d}\alpha B\mathrm{d}\beta &\;=\; M_{\mathrm{applied}}
\end{aligned}
\tag{2.51}
$$

For axially symmetric shells without shearing forces or **Surfaces of Revolution**. Membrane analysis can be applied. Sum forces vertically $R = \int_0^\psi q_a 2\pi r_0 \mathrm{d}\psi = 2\pi a^2 q(1-\cos\psi)$, where $R(\psi)$ is the vertical body force. $R(\psi)+2\pi_{r_0}(N\psi)\sin\psi = 0$. Sum forces inward in a loop. $N_\psi r_0 \mathrm{d}\theta\mathrm{d}\psi + N_\theta r_1 \sin\psi\mathrm{d}\theta\mathrm{d}\psi + q\cos\psi r_0 r_1 \mathrm{d}\psi\mathrm{d}\theta = 0$.

Section forces from stresses. For a uniform stress σ_0 along a section small enough $\mathrm{d}\psi$ such that R can be assumed to be constant. Stress in section $\sigma_0\left(\left(R+\frac{h}{2}\right)^2 - \left(R-\frac{h}{2}\right)^2\right)\frac{\pi\mathrm{d}\psi}{2\pi}$ here h is the thickness of the plate. Stress over midline $N = \frac{\sigma_0}{R\mathrm{d}\psi}\left(R^2 + Rh + \frac{h^2}{4} - R^2 + Rh - \frac{h^2}{4}\right)\frac{\pi\mathrm{d}\psi}{2\pi} = \sigma_0 h$ Notice that shear stresses are only parabolic in the mean– warping in fact does occur. For a thin shell assume:

$$
\sigma_{\max}\left(\pm\frac{h}{2}\right) = \frac{N_{ij}}{h} \pm \frac{6M_i}{h^2} \quad\text{but}\quad \sigma(0) = \frac{3}{2}\frac{Q_0}{h}
\tag{2.52}
$$

For axially symmetric shells without shearing forces or **Surfaces of Revolution**. Membrane analysis can be applied. Sum forces vertically $R = \int_0^\psi q_a 2\pi r_0 \mathrm{d}\psi = 2\pi a^2 q(1-\cos\psi)$, where $R(\psi)$ is the vertical body force. $R(\psi)+2\pi_{r_0}(N\psi)\sin\psi = 0$. Sum forces inward in a loop. $N_\psi r_0 \mathrm{d}\theta\mathrm{d}\psi + N_\theta r_1 \sin\psi\mathrm{d}\theta\mathrm{d}\psi + q\cos\psi r_0 r_1 \mathrm{d}\psi\mathrm{d}\theta = 0$.

2.2.5 Strain and Constitutive Relationship

The **strain displacement** is defined in terms of the change of a length on the surface

$$\epsilon_w = \frac{(r_1 - w)\mathrm{d}\psi - r_1\mathrm{d}\psi}{r\mathrm{d}\psi} = \frac{w}{r} \quad \text{and} \quad \epsilon_v = \frac{\frac{\partial\theta}{\partial\psi}\mathrm{d}\psi}{r\mathrm{d}\psi} = \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}\psi} - \frac{w}{r_1} \qquad (2.53)$$

For linear elastic small displacement the constitutive relationship is given as:

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\psi) \quad \text{and} \quad \epsilon_\psi = \frac{1}{E}(\sigma_\psi - \nu\sigma_\theta) \qquad (2.54)$$

From membrane analysis $\sigma_\theta = N_\theta h$ and $\sigma_\psi = N_\psi h$. Putting these together:

$$\epsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_\psi) \quad \text{and} \quad \epsilon_\psi = \frac{1}{Eh}(N_\theta - \nu N_\theta) \qquad (2.55)$$

2.2.6 Application

The procedure for solving a shell or plate problem is

- Select a coordinate system $\alpha - \beta$

- Test assumptions of thin shell and small deformation

- characteristic lengths and directions

- normal to surface

- curvature (exhibits tensor properties

- equilibrium

- relate section forces to point stresses

- impose constitutive relationship

- find forces and moments in terms of strains and displacements

- impose boundary conditions

- solve

Toroidal Shell

- Determine the coordinate system

$$\begin{aligned}
\vec{r} &= f_1\vec{u} + f_2\vec{v} + f_3\vec{w} \\
f_1 &= (r_1 + r_2\cos\theta_2)\cos\theta_1 \\
f_2 &= (r_1 + r_2\cos\theta_2)\sin\theta_1 \\
f_3 &= r_2\sin\theta_2
\end{aligned} \qquad (2.56)$$

- Determine characteristic lengths A and B

$$\frac{\partial\vec{r}}{\partial\theta_1} = (-(\vec{r}_1 + \vec{r}_2\cos\theta_2)\sin\theta_1)\vec{u} + ((\vec{r}_1 + \vec{r}_2\cos\theta_2)\cos\theta_1)\vec{v} + (0)\vec{w} \quad (2.57)$$

$$A^2 = (r_1 + r_2\cos\theta_2)^2; \quad A = r_1 + r_2\cos\theta_2 \qquad (2.58)$$

$$\frac{\partial\vec{r}}{\partial\theta_2} = \cos\theta_1(-r_2\sin\theta_2)\vec{u} + \sin\theta_1(-r\sin\theta_2)\vec{r} + r_2\cos\theta_2\vec{w} \qquad (2.59)$$

$$B^2 = r_2^2; \quad B = r_2 \qquad (2.60)$$

- determine directions $\vec{t}_\alpha, \vec{t}_\beta, \vec{t}_n$

$$
\begin{aligned}
\vec{t}_\alpha &= \frac{1}{A}\frac{\partial \vec{r}}{\partial \theta_1} = -\sin\theta_1 \vec{u} + \cos\theta \vec{v} \\
\vec{t}_\beta &= \frac{1}{B}\frac{\partial \vec{r}}{\partial \theta_2} = -\cos\theta_1 \sin\theta_2 \vec{u} - \sin\theta_1 \sin\theta_2 \vec{v} + \cos\theta_2 \vec{w} \\
\vec{t}_n &= \vec{t}_\alpha \times \vec{t}_\beta = \cos^2\theta_1 \vec{u} + \sin\theta_1 \cos\theta_2 \vec{v} + \sin\theta_2 \vec{w} \qquad (2.61)
\end{aligned}
$$

- find the curvatures

$$
\frac{1}{R_\alpha} = \frac{1}{A}\vec{t}_n, \alpha \cdot \vec{t}_\alpha = \frac{1}{A}(2\cos\theta_1 \sin^2\theta_1 + \cos^2\theta_1 \cos\theta_2) \qquad (2.62)
$$

analogously

$$
\frac{1}{R_\beta} = \frac{1}{B}\vec{t}_n, \beta \cdot \vec{t}_\beta = \frac{1}{B}(\sin^2\theta_1 \sin\theta_2 \cos\theta_2 + \cos^2\theta_2) \qquad (2.63)
$$

Maximize and minimize to find the principal directions.

Spherical Shell

Pick a spherical coordinate system:

$$
f_1 = r\sin\rho\cos\theta \quad f_2 = r\sin\rho\sin\theta \quad f_3 = r\cos\rho \qquad (2.64)
$$

Thus $\vec{r} = f_1 \vec{u} + f_2 \vec{v} + f_3 \vec{w}$. and the characteristic lengths are $A = r\sin\rho$ and $B = r$. The directions are $\vec{t}_\alpha = \frac{1}{A}\frac{d\vec{r}}{d\theta} = -\sin\theta\vec{u} + \cos\theta\vec{v}$ and $\vec{t}_\beta = \frac{1}{B}\frac{d\vec{r}}{d\theta} = \cos\rho\cos\theta\vec{u} + \cos\rho\sin\theta\vec{v} - \sin\rho\vec{w}$. The normal direction to the shell is $\vec{t}_n = -\cos\theta\sin\rho\vec{u} - \sin\theta\sin\rho\vec{v} - \cos\rho\vec{w}$. Solve for the derivatives.

$$
\begin{Bmatrix}
\vec{t}_{\alpha,\alpha} \\
\vec{t}_{\alpha,\beta} \\
\vec{t}_{\beta,\alpha} \\
\vec{t}_{\beta,\beta} \\
\vec{t}_{n,\alpha} \\
\vec{t}_{n,\beta}
\end{Bmatrix}
=
\begin{bmatrix}
0 & -\cos\rho & -\sin\rho \\
0 & 0 & 0 \\
\cos\rho & 0 & 0 \\
0 & 0 & -1 \\
\sin\rho & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{Bmatrix}
\vec{t}_\alpha \\
\vec{t}_\beta \\
\vec{t}_n
\end{Bmatrix}
\qquad (2.65)
$$

Use membrane analysis to find equilibrium equations:

$$
\frac{d}{d\rho}(N\phi r\sin\rho) - r\cos N_\theta = -r^2\sin^2\rho q \quad \text{and} \quad \frac{N_\theta}{r} + \frac{N_\phi}{r} = -q\cos\rho \qquad (2.66)
$$

Cylindrical bending — pure bending

The curvature of a plate in pure bending without any assumptions beyond the continuum assumptions is derived as follows:
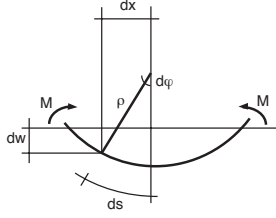
$$\rho \, d\psi = ds \qquad (2.67)$$

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{dx}{ds}\frac{d\psi}{ds} \qquad (2.68)$$

$$\psi = \tan^{-1}\frac{dw}{dx} \qquad (2.69)$$

$$ds^2 = dx^2 + dw^2 \qquad (2.70)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2} \qquad (2.71)$$

The curvature of a plate in pure bending is thus:

$$\chi = \frac{1}{\rho} = \frac{\frac{d}{dx}\left(\tan^{-1}\frac{dw}{dx}\right)}{\sqrt{1 + (\frac{dw}{ds})^2}} = \frac{-\frac{\partial^2 w}{\partial x^2}}{(1 + (\frac{dw}{ds})^2)^{\frac{3}{2}}} \qquad (2.72)$$

For cylindrical shells summing of forces including bending moments results in

$$D\frac{d^4 w}{dx^4} + \frac{Eh}{a^2}w = Z \qquad (2.73)$$

Where $D = \frac{Eh^3}{12(1-v^3)}$ is the bending stiffness.

One dimensional plate behavior— simple beam assumptions

1. neglect shear effect

2. neglect warping: plane sections remain planar

Thus the deformation must be small and $\frac{dw}{ds} \approx 0$
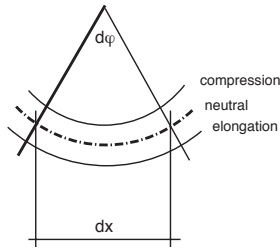
$$\chi_x = -\frac{\partial^2 w}{\partial w^2} \quad \& \quad \chi_x = \frac{d\psi}{dx} \qquad (2.74)$$

$$ds = \rho d\psi \qquad (2.75)$$

$$(\rho + z)d\psi = ds + (u + \frac{\partial u}{\partial x}dx - u) \qquad (2.76)$$

$$zd\psi = \frac{\partial u}{\partial x}dx = \varepsilon_x dx \qquad (2.77)$$

Thus, the **strain** depends on the curvature and the distance from the neutral axis $z\chi_x = \varepsilon_x$. Thus, the sign of the curvature is consistent with the compression $(-)$ and elongation $(+)$ convention.

In cylindrical bending $\chi_y = 0$, so $\varepsilon_y = 0$. And the problem is a plane stress problem $\sigma_z = 0$. The **stress** for a homogenous isotropic plate under pure bending is:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \chi_x z \qquad (2.78)$$

$$\varepsilon_x = (1 - \nu^2)\frac{\sigma_x}{E} \qquad (2.81)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = 0 \qquad (2.79)$$

$$\sigma_x = \frac{E\varepsilon_x}{(1 - \nu^2)} \qquad (2.82)$$

$$\sigma_y = \nu\sigma_x \qquad (2.80)$$

The plate moment then is

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \, \mathrm{d}z = \frac{E}{(1 - \nu^2)}(\chi_x)\int_{-h/2}^{h/2} z^2 \, \mathrm{d}z = \frac{EI}{12(1 - \nu^2)}\chi_x \qquad (2.83)$$

2.3 Reliability

2.3.1 References

- Reliability Based Design in Civil Engineering [11]

- Statistical Analysis for Scientists and Engineers [1]

- Structural Reliability Methods [6]

- Reliability, Prof. Gautam Dasgupta, CU E4225, Fall 1999

2.3.2 Glossary

Probabilistic: Axiomatic Theory

Statistical: Numerical calculations and Estimations

Stochastic: Educated Guessing

Frequency Distributions: Value vs. the frequency of value occurrences.

Mean: First statistical moment. Point at which the peak of the normal distribution occurs. $\bar{x} = \int_{-\infty}^{\infty} x f_x(x)\mathrm{d}x$.

Expected Value: $\mathrm{E}(\widetilde{x}) = \bar{x} = \int_{-\infty}^{\infty} x f_x(x)\mathrm{d}x$

Statistical Moments: In general form

$$(S_\mu)^m = \int_{-\infty}^{\infty} (x - \bar{x})^m f_x(x)\mathrm{d}x = \mathrm{E}[(\widetilde{x} - \mathrm{E}(\widetilde{x}))^m] \qquad (2.84)$$

Standard of Deviation: Second statistical moment. Width of the normal distribution.

Skewness: Third statistical moment. Symmetry of the normal distribution.

Kurtosis: Fourth statistical moment. Behavior of the normal distribution at its ends.

Beta Distribution: β is the beta–function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$.
If $m = n$ the function is symmetric. $f_x = \frac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}$.

Normal Distribution:

$$f_x = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \tag{2.85}$$

Standardized random variable: $\widetilde{z} = \frac{\widetilde{x}-\mu_x}{\sigma_x}$. The mean of $\widetilde{z}$ is zero and the standard of deviation is one.

Limit State: If the value is less than the limit state the system fails otherwise it holds. < 0: failure, > 0: safe, $= 0$: limit state.

2.3.3 Application

The general form of a problem in mechanics is a linear operator L acting upon a response u related to an applied force f.

$$Lu = f \tag{2.86}$$

Any part of the equation may be stochastic.

2.3.4 Theory

Transformation of Distributions

Given $y = g(x)$ the relationship between $f_x(x)$ and $f_y(y)$ can be derived by the relationships between areas. The probability that the actual value lies between x and $x + \mathrm{d}x$ is $f_x(x)\mathrm{d}x$. By equating areas find that: $f_y(y) = \frac{f_x(x)}{\mathrm{dy}/\mathrm{dx}}$. Thus, f_y can be written in terms of x. Notice that areas are being compared so choose only positive values of the $x = g^{-1}(y)$ if there is a choice. For functions whose inverses are single valued:

$$f_y(y) = \frac{f_x(x)}{\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|} \tag{2.87}$$

The linear combination of two uncoupled gaussian distributions is a gaussian distribution.

Statistical Dependence

Use covariance to determine wether or not statistical variables are related.

$$\sigma_{xy}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) = c_{xy} = c_{yx} \tag{2.88}$$

Correlated standard gaussian values ($\bar{x} = 0$ and $\sigma = 1$) can be combined linearly to find uncorrelated values.

$$\widetilde{z}_1 = \alpha_1 \widetilde{x}_1 + \alpha_2 \widetilde{x}_2 \quad (2.89) \qquad\qquad \widetilde{z}_2 = \beta_1 \widetilde{x}_1 + \beta_2 \widetilde{x}_2 \qquad (2.90)$$

For $\widetilde{z}_1$ and $\widetilde{z}_2$ to be uncorrelated the following should be true. Notice that the following calculations assume that $x_1 x_2$, x_1^2, and x_2^2 are uncorrelated.

$$\mathrm{E}(\widetilde{z}_1^2) = \mathrm{E}[(\alpha_1 \widetilde{x}_1 + \alpha_2 \widetilde{x}_2)^2] = \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \; c = 1 \qquad (2.91)$$

$$\mathrm{E}(\widetilde{z}_2^2) = \mathrm{E}[(\beta_1 \widetilde{x}_1 + \beta_2 \widetilde{x}_2)^2] = \beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \; c = 1 \qquad (2.92)$$

$$\begin{aligned} \mathrm{E}(\widetilde{z}_1 * \widetilde{z}_2) &= \mathrm{E}[\alpha_1\beta_1 x_1^2 + (\alpha_2\beta_1 + \beta_2\alpha_1)x_1 x_2 + \alpha_2\beta_2 x_2^2] \\ &= \alpha_1\beta_1 + \alpha_2\beta_2 + (\alpha_2\beta_1 + \beta_2\alpha_1)c = 0 \end{aligned} \qquad (2.93)$$

Reliability Index β

Given a limit state function: $g(\widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots) = f_1(\widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots)$, find the expected value $\mu_g = f_2(\mu_a, \mu_b, \mu_c, \ldots)$. For first order approximations $f_2 = f_1$.

For uncorrelated gaussian random variables:

$$\sigma_g^2 = \sum_i \left(\frac{\partial g}{\partial \widetilde{\eta}_i} \right)^2 \sigma_{\eta_i}^2 = \left(\frac{\partial g}{\partial \widetilde{a}} \right)^2 \sigma_a^2 + \left(\frac{\partial g}{\partial \widetilde{b}} \right)^2 \sigma_b^2 + \left(\frac{\partial g}{\partial \widetilde{c}} \right)^2 \sigma_c^2 + \ldots \qquad (2.94)$$

The variance of g the limit function is $\beta = \frac{\mu_g}{\sigma_g}$.

Let the standardized Gaussian variable be defined as $\widetilde{z} = \frac{\widetilde{D} - \mu_D}{\sigma_D}$, where $D = Resistance - Load$. If $\widetilde{D} < 0$ then $\sigma_D \widetilde{z} + \mu_D < 0$ and $\widetilde{z} < -\frac{\mu_D}{\sigma_D} = -\beta$. Thus, the probability of failure $P_f = \phi(-\beta)$ where ϕ is the cumulative distribution for the standardized Gaussian variable $\widetilde{z}$.

FOSM: First Order Second Moment

FOSM uses a first order taylor expansion and up to the second statistical moment.

Given $[\widetilde{K}]\{\widetilde{u}\} = \{f\}$ define a *reference stiffness* $[K_0] = \mathrm{E}(\widetilde{K})$. Thus the *stiffens deviator* $[\widetilde{\Delta}K] = [\widetilde{K}] - [K_0]$. The *equilibrium equation* is defined as $\{f\} = [K_0]\{U_0\}$. Thus displacement can also be defined in terms of deviation $\{\widetilde{\Delta}u\} = \{\widetilde{u}\} - \{U_0\}$. This equality follows:

$$\{f\} = [\widetilde{K}]\{\widetilde{u}\} = \left([K_0] + [\widetilde{\Delta}K]\right)\left(\{U_0\} + \{\widetilde{\Delta}u\}\right) \qquad (2.95)$$

$$[K_0]\{\widetilde{\Delta}u\} + [\widetilde{\Delta}K]\{U_0\} = 0 \quad \text{so} \quad \{\widetilde{\Delta}u\} = -[K_0]^{-1}[\widetilde{\Delta}K]\{U_0\} \qquad (2.96)$$

This first order expansion is now evaluated. The mean is:

$$\mathrm{E}(\{\widetilde{u}\}) = \mathrm{E}(\{U_0\} + \{\widetilde{\Delta}u\}) = \{U_0\} \qquad (2.97)$$

The covariance is:

$$\mathrm{E}[\{\widetilde{u}\}\{\widetilde{u}\}^T] = [K_0]^{-1}\mathrm{E}\left[[\widetilde{\Delta}K]\left(\{U_0\}\{U_0\}^T\right)[\widetilde{\Delta}K]\right][K_0]^{-1} \qquad (2.98)$$

Thus the covariance of the displacement $\{\widetilde{u}\}$ depends on the covariance of the matrix $[\widetilde{\Delta}K]$.

Bayesian Concepts: Decision Making

Created by Thomas Bayes [1].

For two variables:

$$(A \cap B) = (AB) = (A|B)(B) = (B|A)(A) \tag{2.99}$$

$$(A|B) = \frac{(AB)}{(B)} \qquad (B|A) = \frac{(AB)}{(A)} \tag{2.100}$$

For more variables: by applying eq. 2.99 on regrouped variables:

$$
\begin{aligned}
(A \cap B \cap C) &= ((A \cap B) \cap C) = (C \cap (A \cap B)) \\
&= (C|(A \cap B))\,(A \cap B) = (C|(A \cap B))\,(B|A)(A)
\end{aligned} \tag{2.101}
$$

Notice that the expression is a function of (A). Similarly we can get:

$$(A \cap B \cap C \cap D) = \Big(D\big|(A \cap B \cap C)\Big)\Big(C\big|(A \cap B)\Big)(B|A)(A) \tag{2.102}$$

Thus one can reason from effects to cause. In engineering design-analysis this idea is used in stochastic system identification problems.

Let B_i be a sub event in the set of mutually exclusive B_i, $(i = 1, 2, ...n)$. Given (B_i) and $(A|B_i)$ are given when we want to calculate for one particular r the value of $(B_r|A)$.

$$(B_r|A) = \frac{(A|B_r)(B_r)}{(A)} \qquad (A) = \sum_{i=1}^{n}(A|B_i)(B_i) \tag{2.103}$$

Hence,

$$(B_r|A) = \frac{(A|B_r)(B_r)}{\sum_{i=1}^{n}(A|B_i)(B_i)} \tag{2.104}$$

2.3.5 Application to Systems

If one of the s members of a **statically determinate** system fails then the entire system fails. If for example each element has the same probability of failure $\mathbf{p}_f$ then the structure is only save when all its components are safe. That is

$$P_f = 1 - (1 - \mathbf{p}_f)^s. \tag{2.105}$$

In a **statically indeterminate** system r number of members are redundant. The structure is safe when any r elements fail, or even $r-i$ elements if $0 \le i \le r$. Thus the probability of failure is:

$$P_f = 1 - \sum_{i=0}^{i=r} {}^{s+r}C_{r-1}(1 - \mathbf{p}_f)^{s+i}\mathbf{p}_f^{r-i} \tag{2.106}$$

---

[1] Thomas Bayes, an English clergyman developed the idea in the later half of the eighteenth century.

For simplicity let $k = r - i$ the number of failed pieces, and let $N = s + r$ be the number of members plus the number of redundancies.

$$P_f = 1 - \sum_{k=0}^{k=r} {}^N C_k (1 - \mathbf{p}_f)^{N-k} \mathbf{p}_f^k \qquad (2.107)$$

Now k is just a dummy variable and the sum can be shifted $j = N - k$:

$$P_f = 1 - \sum_{j=s}^{j=N} {}^N C_j (1 - \mathbf{p}_f)^j \mathbf{p}_f^{N-j} \qquad (2.108)$$

Chapter 3

Dynamics and Vibrations

3.1 Linear Vibration

3.1.1 References

- Engineering Vibration [13]
- Vibrations, Prof. Rimas Vacaitis, CU E4215, Fall 1998

3.1.2 Glossary

Steady State: periodic motion whose frequency is independent of time.

Transient: motion is not oscillatory.

Natural Frequency: frequencies at which the system would oscillate if it were vibrating freely. For a single degree of freedom system $w = \sqrt{\frac{k}{m}}$.

Harmonic: sinusoidal periodic motion

    Amplitude: the vertical range of the sine wave is $\pm$ the amplitude.

    Period: the time it takes for the a periodic function to complete its unique component or cycle.

    Frequency: radians per second, or the inverse of the period.

    Radial frequency: cycles per second, or 2π times the frequency.

Average values: $\bar{x} = \lim_{t \to \infty} \int_0^T x(t) \mathrm{d}t$

Mean squared values: $\bar{x}^2 = \lim_{t \to \infty} \int_0^T x^2(t) \mathrm{d}t$

Root Mean Squared: $rms = \sqrt{\bar{x}^2}$

Power: rate of doing work $P = F\frac{\mathrm{d}x}{\mathrm{d}t}$.

Damping: energy dissipation by damping The energy dissipated by damping can be given as $W_d = \oint F_d \mathrm{d}x$. For a single degree of freedom system $W_d = 2\zeta\pi k x^2$. Damping is defined as $c = 2\zeta\sqrt{km}$. For a single degree of freedom system $w_d = w_n\sqrt{1-\zeta^2}$

   Equivalent viscous Damping: for harmonic motion only. The energy dissipated by viscous damping is equal to the energy dissipated by non-viscous damping. This must be evaluated for each particular damping force.

$$\pi c_{eq} w x^2 = W_d \tag{3.1}$$

   Structural Damping: or solid damping. A constant α is found experimentally.

$$\pi c_{eq} w = \alpha \quad \alpha x^2 = W_d \tag{3.2}$$

   Coulomb Damping: from sliding friction. This linear damping is independent of frequency.

$$F_d = F_d(\dot{x}) = \begin{cases} \mu N & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ -\mu N & \dot{x} < 0 \end{cases} \tag{3.3}$$

   Viscous Damping: equivalent to structural damping. Estimate by power or logarithmic method.

Decibel: $db = 10\log_{10}\left(\frac{x_1}{x_0}\right)^2$ where x_0 is the reference amplitude and x_1 is the measured amplitude.

Dunkerley's formula: given a as a flexibility coefficient $[K]^{-1} = [a]$. The sum of the trace of the matrix gives a *lower bound* for the eigenvalue:

$$\det\left([a][M] - \frac{[I]}{w^2}\right) = 0, \quad \sum_{i=1}^{n} a_{ii} m_{ii} = \sum_{i=1}^{n} \frac{1}{w_i^2}, \quad \frac{1}{w_1^2} \leq \sum_{i=1}^{n} a_{ii} m_{ii} \tag{3.4}$$

Rayleigh Method: Energy Method: *upper bound*

$$w^2 \leq \frac{\{u^T\}[K]\{u\}}{\{u^T\}[M]\{u\}} = R(u). \tag{3.5}$$

Guessing a modal displacement $\{u\}$ that makes sense often is a good approximation for the natural frequency w.

Frequency Response Functions: When forcing frequency and natural response frequencies coincide the frequency response increases. In an undamped system it increases to infinity.

$$x(t) = A|G(iw)|\mathrm{e}^{iwt} \quad \text{and} \quad |G(iw)| = \left(\left(1 - \left(\frac{w}{w_n}\right)^2\right)^2 + \left(2\zeta\frac{w}{w_n}\right)^2\right)^{-\frac{1}{2}} \tag{3.6}$$

The phase angle associated with the frequency is:

$$\phi = \tan^{-1}\left(\frac{2\zeta\frac{w}{w_n}}{1 - \frac{w}{w_n}^2}\right) \tag{3.7}$$

3.1.3 Equations of Motion

The equations of motion can be found using conservation of linear momentum or conservation of energy. To apply conservation of momentum a free body diagram is drawn for each mass and the forces about each mass is summed. The energy equation is applied using Lagränges equation where the potential energy $V = \frac{1}{2}kq^2$ and the kinetic energy $T = \frac{1}{2}m\dot{q}^2$. thus for conservative systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial V}{\partial q_k} = Q_k \tag{3.8}$$

A lagrangian $L = T - V$ can be used to reformulate the equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \tag{3.9}$$

Here q are independent generalized coordinates and Q_k are path independent conservative forces. Nonconservative forces such as damping can be included using $F = c_{ij}\dot{q}_i\dot{q}_j$, and add a force $Q_{visc} = \frac{\partial F}{\partial \dot{q}_k}$.

For a continuous distributed parameter system, look at the equilibrium of an infinitesimal piece and sum forces including inertia terms.

3.1.4 General Forced Response

3.1.5 Discrete System

Direct method $\{x(iw)\} = [Z(iw)]^{-1}\{Q_0\}$ where Z is the impedance matrix, and $[Z(iw)] = -w^2[M] + iw[C] + [K]$. Solving

$$\{q(t)\} = \frac{\mathrm{adj}[Z_{iw}]\{Q_0\}e^{iwt}}{\det[Z_{iw}]} \tag{3.10}$$

Modal method For a proportionally damped multi degree of freedom system this method uncouples the system.

- Find eigenvalues $\det([A] - \lambda[I]) = 0$ where $[A] = [M]^{-\frac{1}{2}}[K][M]^{\frac{1}{2}}$.

- Find eigenvectors $([A] - \lambda[I])\{v_i\} = \{0\}$

- Normalize $||v_i|| = \sqrt{\{v_i\}^T\{v_i\}}$ and $\{\bar{v}_i\} = \frac{1}{||v_i||}\{v_i\}$.

- Normalized modes $\{u_i\} = [M]^{-\frac{1}{2}}\{v_i\}$.

- Duhamel's equation

$$\eta_i(t) = \frac{1}{w_{d_i}} \int_0^t N_i(t-\tau)e^{-\zeta_i w_i t} \sin(w_{d_i}\tau)\mathrm{d}\tau. \qquad (3.11)$$

Also, $w_{d_i} = w_i\sqrt{1-\zeta_i^2}$ and $N_i(t) = \{u_i\}^T\{Q(t)\}$. Here $\{Q(t)\}$ is the vector of forcing functions for each degree of freedom.

- Find solution $q(t) = \sum_{i=1}^n \eta_i(t)\{u_i\}$

Harmonic Excitation systems with structural damping exhibit harmonic responses to harmonic input. $[C] = \frac{1}{\pi w[\alpha]}$ also $[\alpha] = \phi\gamma[K]$ and since the system is harmonic $\{\dot{q}\} = iw\{q\}$. The governing equation for such a system is

$$[M]\{\ddot{q}(t)\} + (1+i\gamma)[K]\{q\} = e^{iwt}\{Q_0\} \qquad (3.12)$$

Solve using generalized form:

$$\ddot{\eta}_r(t) + (1+i\gamma)w_r^2 n_r(t) = e^{iwt}N_r \quad \text{where} \quad N_r = \{u_r\}^T\{Q_0\} \qquad (3.13)$$

$$\eta_r(t) = \frac{e^{iwt}N_r}{(1+i\gamma)w_r^2 - w^2} \quad \text{and} \quad \{q(t)\} = \sum_{i=1}^n \frac{e^{iwt}\{u_r\}^T\{Q_0\}}{(1+i\gamma)w_r^2 - w^2}\{u_r\} \qquad (3.14)$$

Shock Spectrum a sudden application of force results in a transient response known as shock.

$$x(t) = \int_0^t F(t)g(t-\tau)\mathrm{d}\tau \quad \text{where} \quad g(t-\tau) = \frac{1}{mw_d}\mathrm{e}^{-\zeta w_n(t-\tau)} \sin w_d(t-\tau) \qquad (3.15)$$

The maximum can be found using the derivative.

Continuous system

Normal Mode Approach to Finite and Continuous Structures Assume small damping and orthogonal modes. the general equation of motion is

$$\mathcal{L}[w] + c(\dot{w}) + m_s(w) = P(\vec{\rho}, t) \qquad (3.16)$$

here $\mathcal{L}$ is the structural operator. Typical values included $-T\frac{\partial^2}{\partial x^2}$ for a taut string, $EI\frac{\partial^4}{\partial x^4}$ for a beam and $D(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2}) + \frac{\partial^4 w}{\partial y^4}$.

The influence response function $h(\vec{r}, \vec{\rho}, t)$ must satisfy the governing equation.

$$\mathcal{L}[h] + c(\dot{h}) + m_s(h) = \delta(\vec{r}-\vec{\rho})\delta t \qquad (3.17)$$

Assume a modal solution

$$h(\vec{r}, \vec{\rho}, t) = \sum_{m=1}^\infty a_m(\vec{\rho}, t)Y_m(\vec{r}) \qquad (3.18)$$

note, each summation is also over the domain. substituting into the equation find:

$$\sum_{m=1}^{\infty} a_m \mathcal{L}[Y_m] + c \sum_{m=1}^{\infty} \dot{a}_m Y_m + m_s \sum_{m=1}^{\infty} \ddot{a}_m Y_m = \delta(\vec{r} - \vec{\rho})\delta t \tag{3.19}$$

First find the value of the normal modes by solving the free vibration solution, notice that the equation can be uncoupled if the nodes are normal so the free modal response of the uncoupled system is of interest.

$$m_2 \ddot{b}_m Y_m + b_m \mathcal{L}[Y_m] = 0 \tag{3.20}$$

For undamped free vibration $\ddot{b}_m = -w_m^2 b_m$. The characteristic condition then is

$$m_s w_m^2 Y_m = \mathcal{L}[Y_m] \tag{3.21}$$

Use orthogonality to find the generalized quantities.

$$\begin{aligned}
\int_R \cdots \int_R m_s Y_m(\vec{r}) Y_n(\vec{r}) \mathrm{d}r &= \begin{cases} M_n & m = n \\ 0 & m \neq n \end{cases} \\
\int_R \cdots \int_R c Y_m(\vec{r}) Y_n(\vec{r}) \mathrm{d}r &= \begin{cases} c_n & m = n \\ 0 & m \neq n \end{cases}
\end{aligned} \tag{3.22}$$

Substitute this into the governing equation.

$$M_m \ddot{a}_m(t, \vec{\rho}) + c_m \dot{a}_m(t, \vec{\rho}) + M_m w_m^2 a_m(t, \vec{\rho}) = Y_m(\rho)\delta(t) \tag{3.23}$$

Assume that $a_m = 0 \ \forall t < 0$. The solution for a_m then by separation of variables is

$$a_m(t, \vec{\rho}) = \begin{cases} Y_m(\vec{\rho}) \exp(-\rho_m w_m t) \sin(w_m^{\Phi} t) & \vec{\rho} \in R \ \& \ t \geq 0 \\ 0 & \vec{\rho} \notin R \text{ or } t < 0 \end{cases} \tag{3.24}$$

where $\rho_m = \frac{c_m}{2M_m w_m}$ and $w_m^{\Phi} = w_m \sqrt{1 - \rho_m^2}$. Also $a_m = Y_m(\vec{\rho}) h_m(t) \ \forall \vec{\rho}, t \in \mathcal{D}$. Where $\mathcal{D}$ is the domain where $\vec{\rho} \in R$ and $t \geq 0$. Now the impulse response function reach mode can be written as:

$$h(\vec{r}, \vec{\rho}, t) = \sum Y_m(\vec{r}) Y_m(\vec{\rho}) h_m(t) \quad \forall \vec{\rho}, t \in \mathcal{D} \tag{3.25}$$

The frequency influence function is found by taking the fourier transform

$$H(\vec{r}, \vec{\rho}, w) = \int_{-\infty}^{\infty} h(\vec{r}, \vec{\rho}, t) e^{iwt} \mathrm{d}t = \sum_{m=1}^{\infty} Y_m(\vec{r}) Y_m(\vec{\rho}) H_m(w) \tag{3.26}$$

Where

$$H_m(w) = \frac{1}{M_m(w_m^2 - w^2 + 2i\zeta_m w w_m)} \tag{3.27}$$

Modal Solution: L is the stiffness operator it is linear homogenous and self-adjoint $(u, Lv) = (Lu, v) = \int_D Lu \cdot v \mathrm{d}D$. Solve partial differential equation.

3.1.6   Vibration Suppression

Frequency comparison

For force isolation problems force and displacement transmissibility must be considered. For a single degree of freed om system the transmissivity ratio for base excitation is:

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta_r)^2}\right]^{\frac{1}{2}} \tag{3.28}$$

for device excitation:

$$\frac{F_t}{KY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1 + r^2)^2 + (2\zeta_r)^2}\right]^{\frac{1}{2}} = r^2 \frac{X}{Y} \tag{3.29}$$

Here $r = \frac{w_{d_r}}{w_n}$ and ζ is the damping ratio.

Displacement: For $r < \sqrt{2}$ the transmissibility ratio is greater than 1. for $r > \sqrt{2}$ the ratio is less than one and the motions are reduced. **Force:** For a single degree of freedom system the displacements are in the form $x(t) = A_0 \cos(w_{d_r} t - \psi)$ and the associated force is $F_T(t) = kx(t) + c\dot{x}(t)$. Here also $r > \sqrt{2}$ the force is decreased.

Vibration Absorbers

Create a two degree of freedom system for steady state vibration. The force transmissivity ratio is a function of the ratio between frequencies of the structure r and the added mass β. A one degree of freedom system becomes a two degree of freedom system with two separate natural frequencies before and after the original natural frequency.

$$\frac{xk}{F_0} = \frac{(2\zeta_r)^2 + (r^2 - \beta^2)^2}{(2\zeta_r)^2(r^2 - 1 + \mu r^2) + (\mu r^2 \beta^2 + (r^2 - (r^2 - \beta^2)))^2} \tag{3.30}$$

Other vibration remedies

include viscoelastic damping treatments and active control solutions using smart materials.

3.2   Nonlinear Vibration

3.2.1   References

- Nonlinear Dynamics

- Dynamics of Systems

- Nonlinear Dynamics

- Nonlinear Dynamical Systems, Prof. Frank Cariello, CU E4101, Spring 2000

3.2.2 Glossary

Linear Equation: An equation which when added and multiplied by scalars remains first order.

Nonlinear Equation: An equation containing higher order quantities.

Phase Space: In general, phase space represents the relationship between n dependent variables of the same independent variables.

1. Phase curves do not intersect for autonomous systems — where unique solutions exist.

2. The integral of motion is a geometric construct

3. Long time behavior can be determined, often without integration.

A system of N dimensions makes N dimensional phase space for an autonomous system. A non-autonomous system is of $N + 1$ dimensions.

Phase Flow: Traces the motion along a phase curve as the independent variable increases.

Phase Portrait: Collection of phase curves which depend on their initial coordinates.

Conservation Laws: For time invariance energy is conserved. For a closed system (static). The vector sum of forces is zero, as is the change in momentum. Momentum itself is conserved.

Every invariance of the lagrangian under some transformation leads to a conserved quantity.

Canonical Transformation: Integration is easier if $\vec{p}, \vec{q} \to \vec{P}, \vec{Q}$ through a canonical transformation such that $\vec{P}$ is chosen to be a constant of motion.

$$\dot{P} = -\frac{\partial H'}{\partial q_i} = 0 \tag{3.31}$$

$$\dot{Q} = \frac{\partial H'}{\partial p_i} = f_i(P_1, \dots, P_n) \tag{3.32}$$

If the Jacobian is constant (one for simplicity) the transformation is canonical.

$$J = \det \begin{bmatrix} P,p & P,q \\ Q,p & Q,q \end{bmatrix} \tag{3.33}$$

Canonical transformations can be generated from the non-uniqueness property of the Lagrangian.

Ergotic Hypothesis:

Functional Dependence: Functions are independent if and only if $F(u_1, u_2) = 0$ here F is an involution.

$$\det\left[\frac{\partial(u_1, u_2)}{\partial(x, y)}\right] = 0 \qquad (3.34)$$

KAM theorem: For a small perturbation most tori are preserved if a (1) super -convergent method is used and (2) the frequencies are sufficiently irrational.

$$\left|\frac{w_1}{w_2} - \frac{r}{s}\right| > \frac{K(\epsilon)}{S^{2.5}} \; \forall \; r, s \qquad (3.35)$$

Continued Fraction Representation: Approximations to r and s can be found with a continued fraction approximation.

$$\sigma = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} \cdots \qquad (3.36)$$

Where a_n is an integer.

Measures of Chaos: One definition of Chaos is a mapping whose behavior is dissipative and whose Liapanov exponent is greater than zero.

Liapanov Exponents: > 0 chaotic < 0 regular.

$$\sigma = \lim_{n \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln\left|\frac{df}{dx}\right|_{x=x_i} \qquad (3.37)$$

also the Liapanov exponents are related to the eigenvalues of the linearized map, thus the are ordered like the eigenvalues are.

$$\sigma_1 = ln|\lambda_1| \qquad (3.38)$$

Dissipative System: A system $\dot{\vec{x}} = \vec{F}(\vec{x})$ is dissipative if $\nabla \cdot \vec{F} < 0$. It is conservative if $\nabla \cdot \vec{F} = 0$. In a dissipative system the sum of the Liapanov exponents is less than one.

Fractals: Fractal dimension is defined by N the number of replicas and r the size of the replica relative to the original.

$$D = \frac{\ln N}{\ln(1/r)} \qquad (3.39)$$

Strange Attractors: Attractors of higher order mappings seem to increase the frequencies in the behavior until they appear chaotic. Actually the frequencies only increase to about order four before chaotic behavior controls. The appearance of these attractors makes the patterns in phase space self replicating.

In general strange attractors occur in a dissipative system of non-integer dimension near to diverging orbits.

3.2.3 Theory

Classical Mechanics

Lagrange

Generalized coordinates: $\vec{x} = \vec{x}(q)$

From Newton's conservative system, $m\ddot{x} = -\frac{\partial V(x)}{\partial x}$ where V is the potential energy.

$$\frac{\partial V(q)}{\partial q} = \frac{\partial V(x)}{\partial x}\frac{\partial x}{\partial q} = -m\ddot{x}\frac{\partial x}{\partial q} = -m\left[\frac{d}{dt}\left[\dot{x}\frac{\partial x}{\partial q} - \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right)\right]\right] \tag{3.40}$$

$$\text{Also}: \quad \frac{\partial x}{\partial q}\dot{q} = \dot{x} \tag{3.41}$$

$$\frac{\partial f(q,\dot{q})}{\partial q} = \left.\frac{\partial f}{\partial q}\right|_{\dot{q}} \quad \& \quad \frac{\partial f}{\partial \dot{q}} = \left.\frac{\partial f}{\partial \dot{q}}\right|_{q} \quad \& \quad \frac{\partial \dot{x}}{\partial q} = \frac{\partial^2 x}{\partial q^2}\dot{q} = \frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) \tag{3.42}$$

$$\frac{\partial V(x)}{q} = -m\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{\dot{x}^2}{2}\right) - \frac{\partial}{\partial q}\left(\frac{\dot{x}^2}{2}\right)\right] \tag{3.43}$$

Recall that Kinetic Energy can be written $T = \frac{1}{2}m\dot{x}^2$.

$$\frac{\partial V}{\partial q} = \frac{-d}{dt}\frac{\partial}{\partial \dot{q}}T + \frac{\partial}{\partial q}T \tag{3.44}$$

Define $L = T - V$ as Lagrange's equation of motion, if $V = V(q)$, then $\frac{\partial V}{\partial \dot{q}} = 0$. Thus:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.45}$$

Every conservative system can be described with n 2nd order equations. All information about the system is contained in the Lagrangian $L(\vec{q},\dot{\vec{q}})$. A benefit to this representation is that energy is not a vector quantity and is consequently invariant to point deformations. The Lagrangian for a system is not unique. $L' = L + \frac{\partial F}{\partial t}(\vec{q},t)$ also satisfies the equation.

Generalized Quantities

E: Constant of motion

$E = \sum q_i \frac{\partial L}{\partial \dot{q}_i} - L$

$\frac{\partial E}{\partial t} = -\frac{\partial L}{\partial t} = 0$ if Lagrange's equation has no explicit time dependence.

$$\frac{d}{dt}L(\vec{q},\dot{\vec{q}},t) = \sum_{i=1}^{N}\frac{\partial L}{\partial q_i}\dot{q}_i + \sum_{i=1}^{N}\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \tag{3.46}$$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{3.47}$$

$$-\frac{dL}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right)\right) \tag{3.48}$$

$F_i = \frac{\partial L}{\partial q_i}$ 　　　　Generalized Force

$P_i = \frac{\partial L}{\partial \dot{q}_i}$ 　　　　Generalized Momentum

Hamiltonian Mechanics

$$H(\vec{p}, \vec{q}, t) = \sum_{i=1}^{N} p_i \dot{q}_i - L(\vec{q}, \dot{\vec{q}}, t), \quad dH = \sum_{i=1}^{N}\left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} q_i\right) - \frac{\partial H}{\partial t} dt \tag{3.49}$$

Using definitions of generalized coordinates, properties of the lagrangian and the definition of the hamiltonian:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad -\frac{\partial H}{\partial q_i} = \dot{p}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3.50}$$

$\vec{p}$, $\vec{q}$ are canonical vectors that form a $2n$ dimensional phase space in which phase trajectories do not intersect and phase volume is preserved.

Geometry of an integrable System

A completely integrable system $H(\vec{p}, \vec{q})$ can be separated into n independent, analytic, single valued first integrals where $\vec{F}(\vec{p}, \vec{q}) \in$ constants. These resulting functions are necessarily independent.

A $2N$ dimensional system can be confined to a N dimensional manifold (Hamiltonian), the geometry is an N dimensional torus. The energy surface is $2N - 1$ dimensional. The ergodic hypothesis is false in general.

Torus

From the Poincaré - Hopf theorem. Each first integral F_m is constrained to a vector field: $\vec{V}_m = (\vec{\nabla}_p F_m - \vec{\nabla}_g F_m)$ with $2n$ components. $\vec{V}_m \perp M$ where M is the integral manifold. $\vec{V}_m \perp \perp M$.

$$\vec{V}_m \cdot (\vec{\nabla}_p F_m, \vec{\nabla}_g F_m) = \{F_m, F_m\} = 0 \tag{3.51}$$

Any compact manifold that is parallelisable with N smooth fields is a N-torus.

Action Angle Coordinates: A torus can be defined by the following action angle coordinates:

I_i 　　Constants of motion

θ_i 　　Periodic, repeats in 2π

$H(\vec{p}, \vec{q}) \to H(I)$ by a canonical transformation. Thus, on the torus $I = \frac{1}{2\pi} \oint p\,dq$, θ increases by 2π in one period. $\dot{I} = \frac{\partial H(I)}{\partial \theta}$ and $\dot{\theta} = \frac{\partial H(I)}{\partial I} = w(I)$ a function of constants.

Motion on a Torus: T_i is the period in the θ_i direction. If the orbits close $n_i T_i = n_j T_j$ where $i = 1, 2, 3, \ldots N$ and $j = 1, 2, 3, \ldots N$. As usual $T_i = \frac{2\pi}{w_i}$. Thus, if $\frac{w_i}{w_j} = \frac{n}{m}$ is a rational number the torus closes, if it is irrational it remains open. The overall period of motion is $T = n_i T_i$ no sum over i where n is an integer.

3.2.4 Methods for Solving

Solving Non-Linear Ordinary Differential Equations

Rewrite any n^{th} order Ordinary Differential Equations (ODE) as a system of n first order ODEs.

Given: $x^{(n)} = F(x^{(n-1)}, x^{(n-2)}, \ldots, x, t)$
Let: $x_1 = x,\ x_2 = \dot{x},\ x_3 = \ddot{x},\ x_n = x^{(n-1)}$
Thus:
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ F(x_n, x_{n-1}, \ldots, x_1, t) \end{pmatrix} \tag{3.52}$$
Here, t is the independent variable, and x is the dependent variable.

For an integrable system: (i) Identify the integral of motion **I**. (ii) Use it to reduce the order of the system. (iii) Solve the resulting integral. (iv) Invert to solve for the single valued function.

Linear Analysis

To find the behavior of a system around the fixed points, linearize it around the fixed points and solve for the behavior.

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \quad \& \quad \vec{x} = \vec{x}^* + \delta\vec{x} \tag{3.53}$$

$$\dot{\vec{x}}^* = \vec{f}(\vec{x}^*) \quad : \text{fixed points} \tag{3.54}$$

$$\delta\dot{\vec{x}} = \nabla \vec{f}(\vec{x}^*)\, \delta\vec{x} \tag{3.55}$$

The solution to this ODE is $\delta\vec{x} = \vec{D}e^{\lambda t}$ where λ are the eigenvalues of $\vec{f}(\vec{x}^*)$.

Case I	$\lambda_2 < \lambda_1 < 0 \ \in \Re$	Stable Node
Case II	$\lambda_2 > \lambda_1 > 0 \ \in \Re$	Unstable Node
Case III	$\lambda_1 < 0 < \lambda_2 \ \in \Re$	Hyperbolic Point
Case IV (a)	$\lambda_{1,2} = \alpha \pm i\beta$	$a < 0$ Stable Spiral
Case IV (b)	$\lambda_{1,2} = \alpha \pm i\beta$	$a > 0$ Unstable Spiral
Case V	$\lambda_{1,2} = iw$	Center or Elliptic Point
Case VI	$\lambda_1 = \lambda_2$	Improper Node
	$\delta\vec{x} = C_1 \vec{D}_1 e^{\lambda_1 t} + C_2(\vec{D}_1 t + \vec{D}_2)e^{-\lambda_1 t}$	
Case VII	$\lambda_1 = \lambda_2 = 0,\ \delta\dot{\vec{x}} = a\delta\vec{x}$	Degenerate Eigenvalues

Nonlinear PDEs

There is little general theory [1]
most equations must be solved specifically.

Soliton Solution: Assume a solution of the form $u(x,t) = f(x - ct)$ so that $u_{,x} = f'$ and $u_{,t} = -cf'$. Solve the soliton wave solution.

Similarity Solution: test for scale invariance of each variable $x_i = k^\alpha x$ solve for the α and find the constant relations. Use these new characteristics to see new ODEs.

Integrals of Motion: In general $\tau_{,t} + \chi_{,x}$ where τ and χ are functions. To use this method, rewrite the PDE in conservation law form. Integrate to the boundaries.

Canonical Perturbation Theory

$$H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta) + \epsilon^2 H_2(I, \theta) \ldots \tag{3.56}$$

To solve: (1) expand all dependent variables in a power series in ϵ, (2) Stipulate for $\epsilon = 0$ get $H(I, \theta) = H_0(I)$. (3) Equate orders of ϵ. (4) Solve the resulting ODEs.

Mappings

The analysis of mappings is similar to the analysis of systems of differential equations.

$$\vec{x}_{n+1} = \vec{T}(\vec{x}_n, t) \tag{3.57}$$

Maps are area preserving if the Jacobian of the mapping function is 1.
The fixed points occur when

$$\vec{x}_* = \vec{T}(\vec{x}_*, t). \tag{3.58}$$

Mappings can also be represented in phase space relative to the eigenvalues of the linearized mapping at the fixed points.

$$\delta x = x_i - x^* \quad \delta y = y_i - y^* \tag{3.59}$$

$$\delta \vec{x}_{i+1} = \begin{pmatrix} f_{,x} & f_{,y} \\ g_{,x} & g_{,y} \end{pmatrix} \delta \vec{x}_i = \lambda \vec{\zeta} \tag{3.60}$$

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[1]**Linear Partial Differential Equations:** PDEs are differential functions of two or more independent variables. General Behaviors of Linear PDEs can be solved by eigenfunction expansion if the boundary conditions are appropriate. Assume a solution of the form $u(x,t) = \sum C_n(t)e^{inx}$. In general boundary conditions and initial conditions need to be given. **Diffusive:** The amplitude decreases in time. **Wave Like:** The amplitude and wave speed remain constant in time. **Dispersive:** The energy of the wave remains constant while the amplitude and wave speed change with time.

The behavior of this area preserving linearized mapping in phase space depends on the eigenvalues.

Case I:	$tr[T] > 2;\ \lambda_1 \& \lambda_2 > 0$ or $\lambda_1 \& \lambda_2 < 0$	-Hyperbolic fixed point
Case II:	$tr[T] < 2;\ \lambda_{1,2} = e^{\pm i\alpha}$	-Elliptic Fixed point
Case III:	$tr[T] = 2;\ \lambda_{1,2} = \pm 1$	-Parabolic Case

Controlling Chaos

If a system depends on a parameter its trajectories can be kept near or on the unstable fixed point. $\vec{x}_{n+1} = F(x_n(p))$ where p is the parameter. First, linearize about the fixed point such that $\delta\vec{x}_n = \widetilde{M}\vec{x}_n$. For $\delta\vec{x}_{n+1}$ to be on the stable manifold choose p sot that $\vec{x}_{n+1}$ is on a stable eigenvector $\vec{\zeta}_s$ of the transform function $\widetilde{M}$. [2]

Expand around the fixed point for the value p.

$$\vec{x}_*(p) = \vec{x}_*(0) + \frac{\partial\vec{x}_*}{\partial p}p + \mathbf{O}(p^2) \tag{3.65}$$

Define $\vec{g} = \frac{\partial\vec{x}_*}{\partial p}$. Thus, $\vec{g}$ is the direction that the fixed point moves as p changes. Then:

$$\vec{x}_{n+1} \approx p\vec{g} + \widetilde{M} \cdot (\vec{x}_n - p\vec{g}) \tag{3.66}$$

3.3 Random Processes

3.3.1 References

- Random Vibrations

- Probability, Random Variables and Stochastic Processes [17]

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[2]**Dyadic Representation** The transform tensor can be written in dyadic form.

$$\widetilde{M} = \lambda_1\vec{\zeta}_1\vec{f}_1 + \lambda_1\vec{\zeta}_2\vec{f}_2 + \ldots \tag{3.61}$$

Where λ are the eigenvalues, $\vec{\zeta}$ are the eigenvectors, and $\vec{f}_2$ are the covariant base vectors. Using the orthogonality of the eigenvectors the values of the base vectors can be found for the given dyadic $\widetilde{M}$.

$$\widetilde{M} \cdot \vec{\zeta}_1 = \lambda_1\vec{\zeta}_1 = (\lambda_1\vec{\zeta}_1\vec{f}_1 + \lambda_1\vec{\zeta}_2\vec{f}_2 + \ldots) \cdot \vec{\zeta}_1 \tag{3.62}$$

thus $f_i\zeta_j = \delta_{ij}$ — the kronecker delta.

Using the dyadic representation. The parameter p must be chosen such that $\vec{x}_{n+1}$ is perpendicular to $\vec{f}_u$. Therefore, set $\vec{f}_u \cdot \vec{x}_{n+1} = 0$.

$$\left[p\vec{g} + (\lambda_u\vec{\zeta}_u\vec{f}_u + \lambda_s\vec{\zeta}_s\vec{f}_s + \ldots) \cdot (\vec{x}_n - p\vec{g}) \right] \cdot \vec{f}_u = 0 \tag{3.63}$$

Here λ_u is an unstable eigenvector while λ_s is stable.

$$p = \frac{\lambda_u}{\lambda_u - 1} \frac{\vec{f}_u \cdot \vec{x}_n}{\vec{f}_u \cdot \vec{g}} \tag{3.64}$$

- Random Processes, Prof. Rimas Vacaitis, CU E6220, Fall 2000

3.3.2   Glossary

Single Random Variables: Variables which are random.

Random Variable (χ): is a function such that for every x both real and complex there exists a probability of $\chi = x$.

Discrete Random Variable: is a variable whose sample space consists of integer values $n \in 1, 2, 3, \ldots$.

Probability Mass Function: the probability that $\chi = x$.

$$0 \leq P_\chi(x) \leq 1 \tag{3.67}$$

$$\sum_{all x} P_\chi(x) = 1 \tag{3.68}$$

Bernouli Distribution: The random variable $\mathcal{K}$ is definded within the sample space: $S = \{0, 1\}$.

$$P_\kappa(k) = \begin{cases} p & k = 1 \\ 1 - p & k = 0 \\ 0 & else \end{cases} \tag{3.69}$$

Poisson Distribution: A random variable defined in the sample space $S = \{0, 1, 2, , 3, \ldots\}$. It may be used to count the number of occurrences of some event in a given time interval $(0, t)$

$$P_N(n) = \begin{cases} \frac{(\lambda t)^n e^{-\lambda t}}{n!} & n = 0, 1, 2, \ldots \\ 0 & else \end{cases} \tag{3.70}$$

Continuous random variable: is defined in the sample space $S \in \Re$.

Probability: the chance that the random variable is a given value $\chi = x$.

$$P(a < x < b) = \int_a^b f_\chi(x) \mathrm{d}x \tag{3.71}$$

Cumulative density function:

$$F_\chi(x) = P_\chi(\chi \leq x) \tag{3.72}$$

$$F_\chi(-\infty) = 0 \quad \text{and} \quad F_\chi(\infty) = 1 \tag{3.73}$$

Probability density function:

$$f_\chi(x) = \lim_{\Delta x \to 0} \frac{F_\chi(x + \Delta x) - F_\chi(x)}{\Delta x} \tag{3.74}$$

$$f_\chi(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_\chi(x) \quad \text{and} \quad \int_{-\infty}^{\infty} f_\chi(x) \mathrm{d}x = 1 \tag{3.75}$$

Uniform: A random variable χ is uniformly distributed over an interval $a \leq x \leq b$.

$$f_\chi(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & else \end{cases} \tag{3.76}$$

Gaussian:

$$f_\chi(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu)^2}{2\sigma_\chi^2}} \tag{3.77}$$

Rayleigh: used for describing the peak values of a random process.

$$f_\chi(x) = \frac{x}{\sigma_x^2} e^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2}; \qquad x > 0 \tag{3.78}$$

Expected Values: Expected values are a calculation of averages. The nth statistical moment is:

$$E[x^n] = \int_{-\infty}^{\infty} x^n f_\chi x \mathrm{d}x \tag{3.79}$$

The first moment is the mean, the second moment is the mean square, the third is skewness and the fourth kurtosis.

Variance:

$$\sigma_\chi^2 = E[(x - \mu_\chi)^2] = E[x^2] - 2\mu_\chi E[x] + \mu_\chi^2 = E[x^2] - \mu_\chi^2 \tag{3.80}$$

Standard of Deviation:

$$\sigma_\chi = \sqrt{E[x^2] - \mu_\chi^2} \tag{3.81}$$

Joint Probability Distribution

Let $X(x)$ and $Y(y)$ be two random variables.

Joint Probability Distribution:

$$f_{XY} = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \tag{3.82}$$

Single distributions can be recovered from the joint distribution.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}x \qquad f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}y \tag{3.83}$$

Cumulative Density Distribution:

$$F_{XY} = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\zeta,\eta)\mathrm{d}\zeta\mathrm{d}\eta \tag{3.84}$$

Normalization condition:

$$F_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(\zeta,\eta)\mathrm{d}\zeta\mathrm{d}\eta = 1 \tag{3.85}$$

Conditional Distributions: The conditional probability density function of X given the outcome of the random variable Y.

$$f_{X|Y}(x|Y_{=y}) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{3.86}$$

Statistical Independence: X and Y are independent.

$$f_{X|Y}(x|Y_{=y}) = f_X(x) \qquad f_{XY} = f_X(x)f_Y(y) \tag{3.87}$$

Joint Statistical Moments:

$$E[X^nY^m] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^m f_{XY}(x,y)\mathrm{d}x\mathrm{d}y \tag{3.88}$$

Covariance:

$$\sigma_{XY} = E[(x-\mu_X)(y-\mu_Y)] = E[XY] - \mu_X\mu_Y \tag{3.89}$$

Correlation Coefficient:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \qquad -1 \le \rho_{XY} \le 1 \tag{3.90}$$

if X and Y are independent then $\rho_{XY} = 0$.

The Bivariate Normal Gaussian:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{XY}^2}}\mathrm{e}^{-\frac{1}{2\left(1-\rho_{XY}^2\right)}\left\{\frac{(x-\mu_X)^2}{\sigma_X^2}-\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right\}} \tag{3.91}$$

Multivariate Jointly Distributed Normal Random Variables:

$$f_{X_1X_2...X_n}(x_1,x_2,\ldots,x_n) = \frac{1}{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left|[S]^{\frac{1}{2}}\right|}\exp\left[-\frac{1}{2}(\vec{x}-\vec{m})^T[S]^{-1}(\vec{x}-\vec{m})\right] \tag{3.92}$$

$S = E[X_iX_j]$: symmetric covariance matrix.
$x_i - m_i = x_i - \mu_i$: normalization.

Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with the means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1, \sigma_2, \ldots \sigma_n$.
Let S_n be the sum of the sequence:

$$S_n = \sum_{i=1}^{n} X_i \qquad M_{sn} = \sum_{i=1}^{n} \mu_i \qquad \sigma_{sn} = \sum_{i=1}^{n} \sigma_i^2 \tag{3.93}$$

As $n \to \infty$ the standardized variable of S_n. By definition the mean is 0 and the variance is 1.

$$Z_n = \frac{S_n - \mu_{sn}}{\sigma_{sn}} \tag{3.94}$$

The standardized variable has the following normal distribution:

$$f_{zn}(z) = \frac{1}{2\pi} e^{-\frac{1}{2}z^2} \quad z \in [-\infty, \infty] \tag{3.95}$$

For any individual distribution of X_i. The distribution of the sum converges to a normalized gaussian distribution.

Random Process: $X(t)$ is an ensemble of time functions that can be characterized statistically. It is a parameterized family of random variables.

$$\{X(t) \in \Omega, t \in T\} \tag{3.96}$$

is a random process defined in state space Ω that evolves with respect to the index set T.

Random processes can be specified by a probability density function of increasing orders of completeness. The nth order is given as below:

$$f_{X(t_1)X(t_2)...X(t_n)}(x_1, x_2, \ldots, x_n) \tag{3.97}$$

Characteristic functions, statistical moment functions and others can be used to characterize random processes.

Strongly Homogenous: the complete set of probability functions is independent of shift of the parametric origin $t_i \to t_i + a$.

Weakly Homogenous: the first two orders of probability functions are independent of a parametric origin shift $t_i \to t_i + a$.

Stationary Process: Homogenous process.

Ensemble Averages: The realizations of random processes $X(t)$ is called the ensemble. Statistical moments can be calculated accordingly.

Mean:
$$\mu_x(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) \mathrm{d}x \tag{3.98}$$

Variance:
$$\sigma_x^2(t) = E[X(t) - \mu_X(t)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_{X(t)} x \mathrm{d}x \tag{3.99}$$

Joint: measure of processes at two times $X(t_1)$ with $X(t_2)$ is the correlation function.

$$R_x(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \tag{3.100}$$

also called the **Autocorrelation Function**. It is always positive definite.

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k R_x(t_1, t_2) \geq 0 \quad \forall a_j \in \Re \tag{3.101}$$

Auto covariance:

$$\begin{aligned} \sigma_{XX}(t_1, t_2) &= E[\{X(t_1) - \mu_X(t_1)X(t_2) - \mu_X(t_2)\}\{x(t_2) - \mu_X(t_2)\}] \\ &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned} \tag{3.102}$$

Autocorrelation Coefficient:

$$\rho_{XY}(t_1, t_2) = \frac{\sigma_{XX}(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} \tag{3.103}$$

Stationary Processes: For a stationary random process the probability structure is invariant to a shift of the time origin. Consequently:

$$\mu_X(t) = \mu_X = \text{const.} \tag{3.104}$$

$$\sigma_X^2(t) = \sigma_X^2 = \text{const.} \tag{3.105}$$

Thus the autocorrelation function depends only on the time change $t_2 - t_1$ not the actual times t_1 and t_2.

$$R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) \quad \text{where} \quad \tau = t_2 - t_1 \tag{3.106}$$

$$R_X(\tau) = R_x(t, t + \tau) = E[X(t) \cdot X(t + \tau)] \tag{3.107}$$

$$\sigma_{XX} = R_X(\tau) - \mu_X \quad \text{and} \quad \rho_{XX}(\tau) = \frac{\sigma_{XX}(\tau)}{\sigma_X^2} \tag{3.108}$$

Properties of correlation functions R_X include:

- $R_X(\tau) = R_X(-\tau)$ —symmetry
- $|R_X(\tau)| \leq R_X(0)$ —boundedness
- $R_X(0) = \sigma_X^2$
- $\lim_{\tau \to \infty} R_X(\tau) = \mu_X$ for non periodic functions. —limit

Examples of Correlation functions include: $R_X(\tau) = \mathrm{e}^{-a|\tau|}$, $R_X(\tau) = \mathrm{e}^{-a\tau^2}$, and $R_X(\tau) = \mathrm{e}^{-a\tau^2} \cos wt$.

Cross-Correlation Function: For two stationary random processes $X(t)$ and $Y(t)$:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \qquad (3.109)$$

This is a function of the combined distribution which is hard to find.

Properties include:

- $R_{XY}(\tau) = R_{YX}(\tau)$ —symmetric
- $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$ —bounded
- $\lim_{\tau \to \infty} R_{XY}(\tau) = \mu_X \mu_Y$ except for periodic —limit

Ergodic Processes: The mean and variance of a stationary ergodic process can be computed using temporal averaging instead of ensemble averaging. Thus, when random process is ergodic in the mean:

$$E[X(t)] = \langle X(t) \rangle = \mu_x = \frac{1}{T} \int_0^T X(t)\mathrm{d}t. \qquad (3.110)$$

This occurs if and only if $E[X(t)] = $ const, $E[X(t)X(t-\tau)] = R_X(\tau)$ and $\lim_{T \to \infty} \int_0^T R_X(\tau)\mathrm{d}\tau = 0$ (except for periodic processes).

When a random process is ergodic in the correlation:

$$
\begin{aligned}
E[X(t)X(t+\tau)] &= R_X(\tau) = \langle X(t)X(t+\tau) \rangle \\
&= \lim_{\tau \to \infty} \frac{1}{T} \int_0^\tau X(t)X(t+\tau)\mathrm{d}t \qquad (3.111)
\end{aligned}
$$

At $\tau = 0$:

$$E[X^2(t)] = \sigma_X^2 = \frac{1}{T} \int_0^T X^2(t)\mathrm{d}t \qquad (3.112)$$

This occurs if and only if: $E[X(t)X(t+\tau)] = f(\tau)$, $S_X(\tau, \mu) = E\{[X(t)X(t+\tau)-R_X(\tau)][X(t+\mu)X(t+\tau+\mu)-R_X(\tau)]\} = g(\tau)$, and $\lim_{t \to \infty} \frac{1}{T} \int_0^T S_X(T, \mu)\mathrm{d}\mu = 0$.

Gaussian Process: A random process for which the joint distribution of

$$X(t_1)X(t_2)\ldots X(t_n) \qquad (3.113)$$

is a joint normal distribution is a Gaussian or normal process. Just as gaussian random variables are completely characterized by their mean and variance, the gaussian random process is completely characterized by its mean and auto correlation functions. The linear combination of a gaussian process is also a gaussian process.

Fourier Transformation: Fourier transformations as applied to random process theory.

Periodic function: $f(t) = f(t + \mathbf{T})$

Fourier Series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(\frac{2\pi nt}{\mathbf{T}}) + b_n \sin(\frac{2\pi nt}{\mathbf{T}}) \right] \tag{3.114}$$

$$a_n = \frac{2}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) \cos(\frac{2\pi nt}{\mathbf{T}}) \mathrm{d}t \quad \text{and} \quad b_n = \frac{2}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) \sin(\frac{2\pi nt}{\mathbf{T}}) \mathrm{d}t$$
$$\tag{3.115}$$

Exponential Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/\mathbf{T}} \quad \text{and} \quad c_n = \frac{1}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) e^{-i2\pi nt/\mathbf{T}} \mathrm{d}t$$
$$\tag{3.116}$$

Fourier Transform As $\mathbf{T} \to \infty$ the Fourier Series degenerates into the Fourier Transform.

$$f(t) = \int_{-\infty}^{\infty} F(w) e^{iwt} \mathrm{d}t \quad \text{and} \quad F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-iwt} \mathrm{d}t \tag{3.117}$$

Piecewise Continuous For a fourier transform to exist a function must be integrable

$$\int_{-\infty}^{\infty} |f(t)| \mathrm{d}t < \infty \tag{3.118}$$

and also, the discontinuities must be finite

$$f(t) = \frac{1}{2}[f(t^+) + f(t^-)] \tag{3.119}$$

Spectral Density Function : Given a stationary random process $X(t)$ the condition $\int_{-\infty}^{\infty} |x(t)| \mathrm{d}t < \infty$ is not satisfied. Thus the fourier transform of the process does not exist. Nevertheless the transform of the correlation function is very useful.

Define the non-existent fourier transform of X as

$$X(w, \mathbf{T}) = \frac{1}{2\pi} \int_{-\mathbf{T}}^{\mathbf{T}} x(t) e^{-iwt} \mathrm{d}t \tag{3.120}$$

with

$$E[X(w, \mathbf{t})X^*(w, \mathbf{t})] = \frac{1}{2\pi}^2 \int_{-\mathbf{T}}^{\mathbf{T}} \int_{-\mathbf{T}}^{\mathbf{T}} R_x(t_1 - t_2) e^{-iw(t_1 - t_2)} \mathrm{d}t_1 \mathrm{d}t_2 \tag{3.121}$$

And

$$E[X(t_1)X^*(t_2)] = R_x(t_1 - t_2) = R_x(\tau) \tag{3.122}$$

Now, the limits t_1 and t_2 are related to the period $\mathbf{T}$. By shifting the period and the area of integration cleverly the result becomes

$$E[|X(w,\mathbf{T})|^2] = \frac{1}{(2\pi)^2} \int_{-2\mathbf{T}}^{2\mathbf{T}} (2\mathbf{T} - |\tau|)R_x(\tau)e^{-iw\tau}\,d\tau \qquad (3.123)$$

Taking the limit as $\mathbf{T} \to \infty$:

$$\lim_{\mathbf{T}\to\infty} \left\{ E[|X(w,\mathbf{T})|^2] \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)e^{-iw\tau}\,d\tau \qquad (3.124)$$

Spectral Density: is defined as this limit.

$$S_{XX}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_X(\tau)e^{-iw\tau}\,d\tau \qquad (3.125)$$

and

$$R_X(\tau) = \int_{-\infty}^{\infty} S_{XX}(w)e^{iw\tau}\,dw \qquad (3.126)$$

Furthermore at $\tau = 0$: $R_X(0) = E[X^2] = \sigma_X^2$

A Spectral density function is both real and even $S_X(w) = S_X(-w)$, $S_{XX} = \frac{1}{\pi} \int_{-\infty}^{\infty} R_X(\tau) \cos w\tau\,d\tau$. Thus $R_X(\tau) = 2 \int_0^{\infty} S_{XX}(w) \cos w\tau\,d\tau$, it is non-negative, and it must decay faster than $\frac{1}{w}$ to be meaningful for a physical process.

Narrow band: a process whose spectral density function is narrow, encompassing a small finite set of frequencies which are adjacent.

Wide band: a process whose spectral density function is wide, encompassing a large finite set of frequencies which are adjacent.

Idealized Gaussian White Noise: A process which encompasses all frequencies – the widest band possible. $S_{XX} = s_0$

Truncated white noise: Bands of frequency are admitted.
Two sided:

$$S_{XX}(w) = \begin{cases} s_0 & w_1 < w < w_2 \\ s_0 & -w_2 < w < w_1 \\ 0 & \text{else} \end{cases} \qquad (3.127)$$

One Sided.

$$S_{XX}(w) = \begin{cases} 2s_0 & w_1 < w < w_2 \\ 0 & \text{else} \end{cases} \qquad (3.128)$$

Unit Gaussian Random Variable:

$$\mu = 0 \quad \sigma_x = 2 \quad P_U(u) = \frac{1}{\sqrt{2\pi}}e^{\frac{-\mu^2}{2}}\,du \qquad (3.129)$$

Thus

$$E[u^n] = \int_{-\infty}^{\infty} \frac{u^n}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\,du = \begin{cases} 0 & n \in \{\text{odd numbers}\} \\ \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right) & n \in \{\text{even numbers}\} \end{cases} \qquad (3.130)$$

Joint Moments of Gaussian Random Variables or Random Processes:
Consider a random variable X_j with $u_j = 0$.

$$E[X_j] = 0 \quad E[X_j^2] = \sigma_{x_j}^2 \quad E[X_j X_k] = \rho_{x_j x_k} \qquad (3.131)$$

$$
\begin{aligned}
E[X_i X_j X_k] &= E[X_k]E[X_i X_j]\ldots = 0 \quad \text{and} \\
E[X_i X_j X_k X_l] &= E[X_i X_j]E[X_k X_l] + E[X_i X_l]E[X_k X_j] + E[X_i X_k]E[X_j X_l] \\
&\quad + E[X_i]E[X_j X_k X_l]\ldots \\
&= \rho_{x_i x_j}\rho_{x_k x_l} + \rho_{x_i x_l}\rho_{x_j x_k} + \rho_{x_i x_k}\rho_{x_j x_l} \qquad (3.132)
\end{aligned}
$$

In general:

$$E[X_1 X_2 \ldots X_{2m+1}] = 0 \qquad (3.133)$$

$$E[X_1 X_2 \ldots X_{2m+1}] = \sum_{\forall \text{comb. } jkrs\ldots} E[X_j X_k]E[X_r X_s] \qquad (3.134)$$

There are N combinations where $N = \frac{(2m)!}{m!2^m}$

3.3.3 Manipulation of Stationary Random Processes

Derivation

Given a stochastic (random) process $X(t)$ its derivative is $\dot{X}(t) = \frac{\mathrm{d}X}{\mathrm{d}t}$. The derivative of the auto correlation function is a cross correlation:

$$\frac{\mathrm{d}}{\mathrm{d}t}R_X(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}E[x(t)X(t+\tau)] = E[X(t)\dot{X}(t+\tau)] = R_{X\dot{X}}(\tau) \qquad (3.135)$$

Similarly, $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) = R_{X\ddot{X}}(\tau)$ again a cross correlation.
The variation τ is arbitrary so it can be rewritten $\mu = t + \tau$ and $t = \mu - \tau$.

$$E[X(t)\dot{X}(t+\tau)] = E[X(t-\tau)\dot{X}(t)] \qquad (3.136)$$

now $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) = -R_{\dot{X}\dot{X}}(\tau)$. which is an auto correlation.
If there are no discontinuities in the auto correlation function $R_{X\dot{X}}(0) = 0$.
Thus $X(t)$ and $\dot{X}(t)$ are uncorrelated or orthogonal at any given time t.

Integration

For a random process $X(t)$

- Converting a random process into a random variable

$$z = \int_a^b X(t)\mathrm{d}t \qquad (3.137)$$

- from random process to random process

$$z(t) = \int_a^t X(s)\mathrm{d}s \quad \text{and} \quad \mu_z(T) = E[z(t)] = \int_a^t \mu_x(s)\mathrm{d}s \qquad (3.138)$$

$$R_{XZ}(t_1 t_2) = \int_a^{t_2} R_{XX}(t_1 s_2)\mathrm{d}s_2 \qquad (3.139)$$

$$R_{XX}(t_1 t_2) = E[z(t_1)z(t_2)] = \int_0^{t_1} \int_0^{t_2} R_{XX}(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2 \qquad (3.140)$$

- Transforming a process

$$z(\eta) = \int_a^b X(t)g(t,\eta)\mathrm{d}t \quad \text{and} \quad \mu_z = \int_a^b X(t)g(t,\eta)\mathrm{d}t \qquad (3.141)$$

$$R_{ZZ}(\eta_1, \eta_2) = \int_a^b \int_a^b R_{XX}(t,s)g(t,\eta)g(s,\eta)\mathrm{d}t\mathrm{d}s \qquad (3.142)$$

Statistical Properties of Random Processes

As the random inputs are transmitted through a linear structural system or filter the output is a narrow band process characterized by that structures response $Y(t)$ and transformed by the frequency response function $H(w)$.

Expected Rate of Threshold Crossing: Let $v(t)$ be a continuous random process with zero mean. Where ζ is a threshold level. Postulate that the structure fails if:

- First Excursion Failure: $Y(t)$ reaches ζ or $-\zeta$ for the first time

- Fatigue Failure: Failure si the result of accumulated damage – a fixed total.

For the second failure postulate, fatigue, counting the number of crossings in a given time period becomes important. Define a random number η that characterizes the number of times $Y(t)$ crosses a threshold level ζ either from above or below.

Crossing times:
$$\eta(\zeta, t_1, t_2) \qquad (3.143)$$

Mean:
$$E[\eta(\zeta, 0, t)] \qquad (3.144)$$

Correlation:
$$E[\eta(\zeta, 0, t_1)\eta(\zeta, 0, t_2)] \qquad (3.145)$$

Let $z[t] = \mathbf{1}[Y(t) - \zeta]$ where $\mathbf{1}$ is the Heaviside Step function.

$$z(t) = \begin{cases} 1 & Y(t) > zeta \\ \frac{1}{2} & Y(t) = \zeta \\ 0 & Y(t) < \zeta \end{cases} \tag{3.146}$$

Using this definition

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{1}[Y(t) - \zeta] = \delta[Y(t) - \zeta] \dot{Y}(t) \tag{3.147}$$

Notice that $z[t]$ is random.

$$\eta(\zeta, t_1, t_2) = \int_{t_1}^{t_2} |\dot{z}(t)| \mathrm{d}t = \int_{t_1}^{t_2} |\dot{Y}(t) \delta[Y(t) - \zeta]| \mathrm{d}t \tag{3.148}$$

$$E[\eta(\zeta, t_1, t_2)] = \int_{t_1}^{t_2} E\{|\dot{Y}(t) \delta[Y(t) - \zeta]|\} \mathrm{d}t = \int_{t_1}^{t_2} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{y}| \delta(y - \zeta) f_{Y\dot{Y}}(y, \dot{y}) \mathrm{d}y \mathrm{d}\dot{y} \right] \mathrm{d}t \tag{3.149}$$

$$E[\eta(\zeta, t_1, t_2)] = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} |\dot{y}| f_{y, \dot{y}}(\zeta, \dot{y}) \mathrm{d}\dot{y} \mathrm{d}t \tag{3.150}$$

Determine the rate of threshold crossing per unit time $N(\zeta, t)$:

$$N(\zeta, t) = \int_{-\infty}^{\infty} |\dot{y}| f_{Y\dot{Y}}(\zeta, \dot{y}) \mathrm{d}\dot{y} \tag{3.151}$$

Similarly the correlation function of $N(\zeta, t)$

$$\phi_{NN}(\zeta, t_1, t_2) = E[N(\zeta, t_1) N(\zeta, t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{y}_1| |\dot{y}_2| f_{Y_1 Y_2}(\zeta, \dot{y}_1, \dot{y}_2) \mathrm{d}\dot{y}_1 \mathrm{d}\dot{y}_2 \tag{3.152}$$

To count only upcrossings of $zeta$

$$E[N_+(\zeta, t)] = \int_0^{\infty} \dot{y} f_{Y\dot{Y}}(\zeta, y) \mathrm{d}\dot{y} \tag{3.153}$$

For a stationary random process

$$E[N_+(\zeta, t)] = \frac{1}{2} E[N(\zeta, t)] \tag{3.154}$$

Upcrossings of Gaussian Random Process

$$N_+(\zeta, t) = N_+(\zeta) \quad \text{Stationary Random Process} \tag{3.155}$$

$$E[N_+(\zeta)] = \int_0^{\infty} \dot{y} \frac{1}{2\pi \sigma_y \sigma_{\dot{y}}} e^{-\frac{1}{2}(\frac{\zeta^2}{\sigma_y^2} + \frac{\dot{y}^2}{\sigma_{\dot{y}}^2})} \mathrm{d}\dot{y} = \frac{1}{2\pi \sigma_y \sigma_{\dot{y}}} e^{\frac{-\zeta^2}{2\sigma_y^2}} \int_0^{\infty} \dot{y} e^{-\frac{\dot{y}^2}{2\sigma_{\dot{y}}^2}} \mathrm{d}\dot{y} \tag{3.156}$$

Let $\mu = \frac{\dot{y}}{2\sigma_{\dot{y}}^2}$, and $\mathrm{d}u = \frac{\dot{y}}{\sigma_{\dot{y}}^2}\mathrm{d}y$

$$E[N_+(\zeta)] = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y}e^{\frac{-\zeta^2}{2\sigma_{\dot{y}}^2}} \tag{3.157}$$

Since the variances depend on the spectral density. If the spectral density is known the upcrossing rate can be found. The zero upcrossing rate occurs at $\zeta = 0$

$$E[N_+(0)] = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y} = \frac{1}{2\pi}\left[\frac{\int_{-\infty}^{\infty}w^2 S_{YY}(w)\mathrm{d}w}{\int_{-\infty}^{\infty}S_{YY}(w)\mathrm{d}w}\right] \tag{3.158}$$

For a narrow band process:

$$E[N_+(0)] = f_e = \frac{1}{2\pi}\sqrt{\frac{2S_0 w_0^2 \Delta w}{2\S_0 \Delta w}} = \frac{w_0}{2\pi} \tag{3.159}$$

For a gaussian process: as variance increases the change for upcrossings decreases:

$$\frac{N_+(\zeta)}{N_+(0)} = e^{\frac{-\zeta^2}{2\sigma_{\dot{y}}^2}} \tag{3.160}$$

The Rate of Peaks

Peak: occurs in a sample function $Y(t)$ when $\dot{Y}(t) = 0$ and $\ddot{Y}(t) < 0$

Valley: occurs in a sample function $Y(t)$ when $\dot{Y}(t) = 0$ and $\ddot{Y}(t) > 0$

A Peak Rate [Middleton, Rice].

$$P_R(t) = |\ddot{Y}(t)|\delta[\dot{Y}(t)]\mathbf{1}[Y(t) - \zeta]. \tag{3.161}$$

The number of peaks in time $t \in (t_1, t_2)$

$$\epsilon[\zeta, t_1, t_2] = \int_{t_1}^{t_2} P_R(t)\mathrm{d}t. \tag{3.162}$$

The expected total number of peaks in a time interval

$$E[\epsilon(\zeta, t_1, t_2)] = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\ddot{y}|\delta|\dot{y}|\mathbf{1}(y-\zeta)f_{Y\dot{Y}\ddot{Y}}(y,\dot{y},\ddot{y})\mathrm{d}y\mathrm{d}\dot{y}\mathrm{d}\ddot{y}. \tag{3.163}$$

Define $M[\zeta, t]$ as the number of peaks above ζ per unit time

$$\epsilon(\zeta, t_1, t_2) = \int_{t_1}^{t_2} M[\zeta, t]\mathrm{d}t. \tag{3.164}$$

The expected peak rate:

$$E[M(\zeta, t)] = -\int_{\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\dot{y}\int_{-\infty}^{0}f_{Y\dot{Y}\ddot{Y}}(y,\dot{y},\ddot{y})\delta\dot{y}\mathbf{1}(y-\zeta)\mathrm{d}\ddot{y}$$

$$= -\int_{\zeta}^{\infty}\int_{-\infty}^{0} f_{Y\dot{Y}\ddot{Y}}(y,0,\ddot{y})\ddot{y}\mathrm{d}y\mathrm{d}\ddot{y}. \tag{3.165}$$

To get a total number of points let $\zeta \to \infty$

$$E[M_T(t)] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y\dot{Y}\ddot{Y}}(y,0,\ddot{y})\ddot{y}\mathrm{d}y\mathrm{d}\ddot{y}. \tag{3.166}$$

If $Y(t)$ is a stationary and gaussian process:

$$f_{Y\dot{Y}\ddot{Y}}(y,0,\ddot{y}) = \frac{1}{(2\pi)^{\frac{3}{2}}|\Lambda|}\exp\left\{-\frac{1}{2|\Lambda|}(\sigma_2^2\sigma_3^2 y^2 + 2\sigma_2^4 y\ddot{y} + \sigma_1^2\sigma_2^2\ddot{y}^2)\right\}. \tag{3.167}$$

Here, $\Lambda = \sigma_2^2(\sigma_1^2\sigma_3^2-\sigma_2^4)$, $\sigma_1^2 = \sigma_y^2 = \int_{-\infty}^{\infty} S_{YY}(w)\mathrm{d}w$, $\sigma_2^2 = \sigma_{\dot{y}}^2 = \int_{-\infty}^{\infty} wS_{YY}(w)\mathrm{d}w$, and $\sigma_3^2 = \sigma_{\ddot{y}}^2 \int_{-\infty}^{\infty} w^2 S_{YY}(w)\mathrm{d}w$.
  Substitute values and integrate to find:

$$E[M_T(t)] = E[M_T] = \frac{1}{2\pi}\frac{\sigma_3}{\sigma_2} = \frac{1}{2\pi}\frac{\int_{-\infty}^{\infty} w^4 S_{YY}\mathrm{d}w}{\int_{-\infty}^{\infty} w^2 S_{YY}\mathrm{d}w}. \tag{3.168}$$

The Probability Density of a Peak Distribution The conditional probability distribution function of peak magnitude at time t is equal or than less than ζ.

$$F_I(\zeta,t)\left[1 - \frac{E[M(\zeta,t)]}{E[M_T(t)]}\right] \tag{3.169}$$

Differentiate to find the probability density function:

$$f_I(\zeta,t) = \frac{\partial}{\partial z}F(\zeta,t) = -\frac{1}{E[M_T(t)]}\frac{\partial}{\partial\zeta}\{E[M(\zeta,t)]\}. \tag{3.170}$$

Using the Leibnitz rule of differentiation:

$$F_I(\zeta,t) = -\frac{1}{E[M_T(t)]}\int_{-\infty}^{\infty}\ddot{y}f_{Y\dot{Y}\ddot{Y}}(\zeta,0,\ddot{y})\mathrm{d}\dot{y}. \tag{3.171}$$

For a stationary random process $f_I(\zeta,t) = f_I(\zeta)$. Assume $Y(t)$ is a Gaussian random process substituting equations given and integrating:

$$\begin{aligned}
f_I(\zeta) &= \frac{(1-\alpha^2)^{\frac{1}{2}}}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{\zeta^2}{2\sigma_1^2(1-\alpha)^2}\right\} \\
&\quad + \frac{\alpha\zeta}{2\sigma_1}\left\{1 + \mathrm{erf}\left[\frac{\zeta}{\sigma_1}(2\alpha^{-2}-2)^{-\frac{1}{2}}\right]\right\}\exp\left(-\frac{\zeta^2}{2\sigma_1^2}\right) \tag{3.172}
\end{aligned}$$

where $\alpha = \frac{E[N_+(0)]}{E[M_T]} = \frac{\sigma_2^2}{\sigma_2\sigma_3}$. α ranges from 0 to 1. If $\alpha = 0$ for the most part the process fluctuates above or below zero, since it is stationary some crossings do occur. If $\alpha = 1$ the process behaves like a narrow band process, one crossing per fluctuation.

$$f_I(\zeta) \approx \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\zeta^2 2\sigma_1^2} \quad \text{for} \quad \alpha = 0: \quad \text{Gaussian}$$

$$f_I(\zeta) \quad = \quad \frac{\zeta}{\sigma_1} \mathrm{e}^{\frac{-\zeta^2}{2\sigma_1^2}} \quad \text{for} \quad \alpha = 0: \qquad \text{Rayleigh} \qquad (3.173)$$

Useful Approximations If reversals can be neglected $E[M(\zeta)] \approx E[N_+(\zeta)]$ and $E[M_T] \approx E[N_+(0)]$.

$$F_I(\zeta) = -\frac{1}{E[N_+(0)]} \frac{\mathrm{d}}{\mathrm{d}\zeta} \int_0^\infty \dot{y} f_{Y\dot{Y}}(\zeta, \dot{y}) \mathrm{d}y. \qquad (3.174)$$

For a gaussian random process $E[N_+(0)] = \frac{1}{2\pi} \frac{\sigma_{\dot{Y}}}{\sigma_Y}$. Thus, if an upcrossing problem is solved the peak crossing problem is solved automatically. Thus for a gaussian process sit can be shown quickly that the narrow band peak crossing distribution is a Rayleigh distribution.

3.3.4 Application and Design

Functions of Random Variables:

$Y = g(X)$ since X is a random variable Y is also a random variable. The statistical properties of Y depend on X.

For functions whose inverse are also functions.

$$f_Y(y) = f_X(x) \left| \frac{\partial x}{\partial y} \right| \quad a \le x \le b, \ \min g(x) \le y \max g(x) \qquad (3.175)$$

For functions whose inverse have n mappings.

$$f_Y(y) = n \times f_X(x) \left| \frac{\partial x}{\partial y} \right| \qquad (3.176)$$

Single Degree of Freedom System

- **Equation of motion**: $m\ddot{x} + c\dot{x} + kx = f(t)$ or $m\ddot{x} + c(\dot{x} - x_0) + k(x - x_0) = 0$. In standard form $y = x - x_0$ and $m\ddot{y} + c\dot{y} + ky = -m\ddot{x}_0 = f(t)$

  For convenience let $w_0 = \sqrt{\frac{k}{m}}$, $c_{cr} = 2\sqrt{km}$ and $\zeta = \frac{c}{c_{cr}}$. Thus the equation of motion can be written generally as:

  $$\ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = \frac{f(t)}{m}. \qquad (3.177)$$

- **Frequency Response Method**: $f(t) = F_0 \mathrm{e}^{iwt}$. The equation can be solved as $x(t) = F_0 H(w) \mathrm{e}^{iwt}$. Where the frequency response function is $H(w) = \frac{1}{m[w_0^2 - w^2 + 2i\zeta w_0 w]}$. Often the frequency response function.

  $$|H(w)|^2 = \frac{1}{m^2[(w_0^2 - w^2)^2 + 4\zeta^2 w_0^2 w^2]}. \qquad (3.178)$$

- **Arbitrary Input** Impulse Response function $f(t)$ is arbitrary. The solution can be found using Duhamel's integral:

$$x(t) = Ce^{-(\zeta + i\sqrt{1-\zeta^2})wt} + \int_0^t f(\tau)h(t-\tau)d\tau. \qquad (3.179)$$

The impulse response function

$$h(t-\tau) = \begin{cases} \dfrac{1}{mw_0\sqrt{1-\zeta^2}}\exp(-\zeta w_0 t)\sin(\sqrt{1-\zeta^2}w_0 t) & t > 0 \\ \qquad\qquad 0 & t < 0 \end{cases} \qquad (3.180)$$

The frequency response function and impulse response function are fourier transform pairs.

$$H(w) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{iwt}dt \quad \text{and} \quad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(w)e^{iwt}dw. \qquad (3.181)$$

Response of a Single Degree of Freedom System to Random Excitation The same equation of motion governs a system with random excitation. If the input is random the output is also random. The solution can be found in the time domain using Duhamel's integral

$$X(t) = De^{(-\zeta + i\sqrt{1-\zeta^2})w_0 t} + \int_0^t F(\tau)h(t-\tau)d\tau \qquad (3.182)$$

Assume that the effect of the initial condition has died down and the resulting random response is $X(t) = \int_0^t F(\tau)h(t-\tau)d\tau$. The expected values of the random variable $X(t)$ can be describe the random variable.

$$\begin{aligned} E[X(t)] &= \int_0^t E[F(\tau)]h(t-\tau)d\tau \\ E[X(t_1)X(t_2)] &= \int_0^{t_1}\int_0^{t_2} E[F(\tau_1)F(\tau_2)]h(t-\tau_1)h(t-\tau_2)d\tau_1 d\tau_2 \\ &\vdots \\ E[X(t_1)X(t_2)\cdots X(t_n)] &= \int_0^{t_1}\cdots\int_0^{t_n} E[F(\tau_1)F(\tau_2)\cdots F(\tau_n)]h(t_1-\tau_1) \\ &\qquad \cdots h(t_2-\tau_2)h(t_n-\tau_n)d\tau_1 d\tau_2\cdots d\tau_n \qquad (3.183) \end{aligned}$$

If the forcing function $F(t)$ is a *Gaussian Random Process* only the first and second moments are needed. The mean $E[F(\tau)] = \mu_F$ is constant for stationary processes. $E[X(t)] = \mu_F \int_0^t h(\tau)d\tau$, for $t > 0$. As $t \to \infty$ the integral $\int_0^t h(\tau)d\tau = H(0)$. Thus, $\mu_x = \mu_F H(0) = \frac{\mu_F}{k}$ since $H(0) = \frac{1}{mw_0^2} = \frac{1}{k}$. Usually random processes are constructed such that $\mu_F = 0$

Stationary and Gaussian Excitation The response is also gaussian if the structure is linear. If the mean is assumed to be zero then $E[F(\tau_1)F(\tau_2)] =$

$R_{FF}(\tau_1 - \tau_2)$. The spectral density is related to the autocorrelation function $R_{FF}(\tau_1 - \tau_2) = \int_{-\infty}^{\infty} \Phi_{FF}(w) e^{iw(\tau_1 - \tau_2)} dw$. For convenience set $E[X(t_1)X(t_2)] = \Phi_{XX}(t_1, t_2)$.

$$\Phi_{XX}(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} d\tau_1 d\tau_2 \int_{-\infty}^{\infty} dw \Phi_{FF}(w) e^{iw(\tau_1 - \tau_2)} h(t - \tau_1) h(t - \tau_2)$$

(3.184)

Integrate over τ_1 and τ_2 to find

$$\Phi_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(w) \mathbf{H}(w, t_1) \mathbf{H}^*(w, t_2) e^{iw(t_1 - t_2)} dw \qquad (3.185)$$

where $\mathbf{H}(w, t) = \int_0^t h(u) e^{iwu} du$, $h(u)$ is the impulse response function.

$$\mathbf{H}(w, t) = \int_0^t \frac{1}{mw_d} e^{\zeta w_0 t} \sin(w_d t) e^{-iwt} dt \quad \text{where} \quad w_d = w_0 \sqrt{1 - \zeta^2} \quad (3.186)$$

$$\mathbf{H}(w, t) = H(w)(1 - (\cos(w_d t) + \frac{\zeta w_0 + iw}{w_d} \sin(w_d t)) e^{(-\zeta w_0 + iw)t}) \qquad (3.187)$$

Taking the limit

$$\lim_{t \to \infty} \mathbf{H}(w, t) = H(w) \qquad (3.188)$$

Using these equations the response correlation function can be written in terms of $H(w)$ but the expression is quite lengthy. The means square can be found similarly and is also lengthy.

Notice, (1) at the beginning the response is non-stationary due to a stationary input, (2) for a large t_1 and t_2 the response is stationary.

$$\Phi_{XX}(t_1, t_2) \to R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{FF}(w) |H(w)|^2 e^{iw\tau} dw \qquad (3.189)$$

$$E[X^2(t)] \to R_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{FF}(w) |H(w)|^2 dw \qquad (3.190)$$

(3) At $t = 0$, $E[X^2(t)] = 0$, assume the system is initially at rest,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(w) e^{iw\tau} dw, \quad \text{thus} \quad S_{XX}(w) = \Phi_{FF}(w) |H(w)|^2. \quad (3.191)$$

Now enough information exists to find all the information about a single degree of freedom process, after a long period of time.

For example, consider idealized white noise.

$$R_{FF}(\tau) = 2\pi S_0 \delta(\tau) \quad \text{and} \quad \Phi_{FF}(w) = S_0 \qquad (3.192)$$

The bandwidth at half power can be used to describe the damping $2\zeta w_0 = \Delta w_{\frac{1}{2}}$. The fundamental damping coefficient is calculated from the fundamental frequency.

$$R_{XX}(0) = \sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(w) dw = \int_{-\infty}^{\infty} \Phi_{FF}(w) |H(w)|^2 dw \qquad (3.193)$$

For idealized white noise

$$\sigma_X^2 = \frac{s_0}{m^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{(w_0^2 - w^2)^2 - 4\zeta^2 w_0^2 w^2}. \tag{3.194}$$

This can be solved by residue theory. $\sigma_X^2 = \frac{S_0 \pi}{2\zeta m^2 w_0^3}$. The gaussian white noise idealization is especially applicable when damping is light. It is not suitable for all problems. When the idealization is valid

$$R_{XX} = \frac{\pi \Phi_{FF}(w_0)}{2m^2 \zeta w_0^3} \tag{3.195}$$

Therefore, a one degree of freedom system can be solved either in time domain from the time history:

$$\mu_X = \bar{X} = \frac{1}{T} \int_0^T x(t)\mathrm{d}t \approx \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$E[X^2] = \frac{1}{T} \int_0^T x^2(t)\mathrm{d}t \approx \frac{1}{N-1} \sum_{i=1}^{N} x_i^2 \tag{3.196}$$

Now $\sigma_X = \sqrt{E[x^2] - \mu_X^2}$ and $\sigma_{\dot{X}} = \sqrt{E[\dot{x}^2] - \mu_{\dot{X}}^2}$. Take the derivative of the time domain numerically. Thus for a gaussian process:

$$
\begin{aligned}
f_F(z) &= \frac{1}{\sqrt{2\pi\sigma_F^2}} e^{\frac{-(z-\mu_F)^2}{2\sigma_F^2}} & \text{forcing function} \\[2mm]
f_X(z) &= \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(z-\mu_F)^2}{2\sigma_X^2}} & \text{response } x \\[2mm]
E[N_+(\zeta)] &= \frac{1}{2\pi} \frac{\dot{\sigma}_X}{\sigma_X} e^{-\frac{\zeta^2}{2\sigma_X^2}} & \text{upcrossing rate} \\[2mm]
f_I(\zeta) &= \frac{\zeta}{\sigma_X^2} e^{\frac{-\zeta^2}{2\sigma_X^2}} & \text{peak rate}
\end{aligned} \tag{3.197}
$$

Or it can be solved from frequency domain. First solve for the static response, and then the dynamic:

$$
\begin{aligned}
S_{XX}(w) &= \Phi_{FF}(w)|H(w)|^2 \\
S_{\dot{X}\dot{X}}(w) &= w^2 \Phi_{FF}(w)|H(w)|^2 \\
\sigma_X &= \sqrt{\frac{\Phi_{FF}(w_0)\pi}{2m^2\zeta w_0^3}} \\
\sigma_{\dot{X}} &= \sqrt{\frac{\Phi_{FF}(w_0)\pi}{2m^2\zeta w_0}}
\end{aligned} \tag{3.198}
$$

Forcing function, upcrossing and peak rate can be solved as before.

Multi-Degree of Freedom System

Assume the system is linear, the inputs are stationary, ergodic and Gaussian. Analysis can be performed either by the direct method or by uncoupling the equation using normal modes.

The general governing equation in terms of the mass matrix $[M]$ the stiffness matrix $[K]$ the damping matrix $[C]$ and the forcing function $\vec{F}(t)$ is:

$$[M]\ddot{\vec{x}} + [K]\vec{x} + [C]\dot{\vec{x}} = \vec{F}(t). \tag{3.199}$$

Frequency and impulse response functions The frequency response function and impulse response function can also be given as a matrix.

$$\vec{x} = [H(w)]\mathrm{e}^{iwt}\mathbf{1} \quad \text{if} \quad F = \mathbf{1}\mathrm{e}^{iwt} \quad \text{where} \quad \mathbf{1} = \left\{ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right\} \tag{3.200}$$

Substituting into the equation of motion

$$[H(w)] = (-w^2[M] + iw[c] + [k])^{-1}. \tag{3.201}$$

The impulse response function and frequency response function are Fourier Transform pairs.

The Direct Method

- Given a any reasonably well behaved forcing function, the response can be found as:

$$\{x(t)\} = \int_{-\infty}^{\infty} [h(\theta)]\{F(t - \theta)\}\mathrm{d}\theta \tag{3.202}$$

- The response at a particular location j

$$X_j(t) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} h_{jk}(\theta) F_k(t - \theta)\mathrm{d}\theta \tag{3.203}$$

- The mean value response at that location then is:

$$\mu_{x_j} = E[X_j(t)] = \sum_{k=1}^{n} \int_{-\infty}^{\infty} h_{jk}(\theta) E[F_k(t - \theta)]\mathrm{d}\theta \tag{3.204}$$

- Since the forcing function is a stationary process, its mean is constant

$$\mu_{xj} = \sum_{k=1}^{n} \mu_{Fk} \int_{-\infty}^{\infty} h_{ik}(\theta)\mathrm{d}\theta = \sum_{k=1}^{n} \mu_{Fk} H_{jk}(0) \quad \{\mu_x\} = [H(0)]\{\mu_f\}. \tag{3.205}$$

- The autocorrelation function is formulated similarly

$$
\begin{aligned}
[R_x(\tau)] &= E[\{X(t)\} \cdot \{X(t-\tau)\}^T] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [h(\theta_1)] E[\{F(t-\theta_1)\}\{F(t+\tau-\theta_2)\}^T][h(\theta_2)] \mathrm{d}\theta_1 \mathrm{d}\theta_2
\end{aligned}
$$

$$(3.206)$$

- No the spectral density response can be found since

$$
[S_x(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [R_x(\tau)] \mathrm{e}^{iw\tau} \mathrm{d}\tau, \quad [S_F(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [R_F(\tau)] \mathrm{e}^{iw\tau} \mathrm{d}\tau
$$

$$(3.207)$$

Combining the response spectral density definitions with the autocorrelation function:

$$[S_x(w)] = [H(w)][S_F(w)][H^*(w)]^T \tag{3.208}$$

Spectral densities and cross spectral densities are included in this formulation.

- The variance and covariance (for a zero mean)

$$\sigma_{x_i}^2 = R_{x_i}(0) = \int_{-\infty}^{\infty} S_{x_i x_i}(w) \mathrm{d}w \quad \text{and} \quad \sigma_{x_i x_j} \int_{-\infty}^{\infty} S_{x_i x_j}(w) \mathrm{d}w. \tag{3.209}$$

Normal Mode Method Solve the harmonic undamped system

$$[M]\{\ddot{x}\} + [k]\{x\} = 0 \quad \text{let } \{x\} = \{\psi\}\mathrm{e}^{iwt} \tag{3.210}$$

Solve the eigenvalue problem to find the natural frequencies and modes.

   Modal response Let $[\Psi]$ be the matrix comprised of all coordinate vectors $\{\psi\}$. Let $\{x\} = [\Psi]\{Y\}$, where Y are the generalized coordinates. Substituting into the governing equation:

$$[M][\Psi]\{\ddot{Y}\} + [C][\Psi]\{\dot{Y}\} + [k][\Psi]\{Y\} = F \tag{3.211}$$

Multiply by the transpose of the eigenvector matrix to uncouple the equation

$$[\Psi]^T[M][\Psi]\{\ddot{Y}\} + [\Psi]^T[C][\Psi]\{\dot{Y}\} + [\Psi]^T[k][\Psi]\{Y\} = [\Psi]^T F \tag{3.212}$$

Now the generalized mass M_i, generalized stiffness, generalized (structural) damping, and generalized force can be formulated:

$$\ddot{Y}_j + 2\zeta_j w_j \dot{Y}_j + w_j^2 Y_j = F_j \tag{3.213}$$

The solution for each mode are then $Y_j(t) = \int_{-\infty}^{\infty} h_j(\tau) F_j(t-\tau) \mathrm{d}\tau$. If all the generalized responses are know the final response can be recalculated using the modal matrix. Or statistical properties can be found for each mode and combined by some scheme which presents an adequate approximation.

Linear Continuous System

$$
\text{input} \rightarrow \boxed{\begin{array}{l} \text{Structure} \\ \cdot \text{ Impulse influence function } h(\vec{r}, \vec{\rho}, t) \\ \cdot \text{ Frequency influence function } H(\vec{r}, \vec{\rho}, w) \end{array}} \rightarrow \text{output}
$$
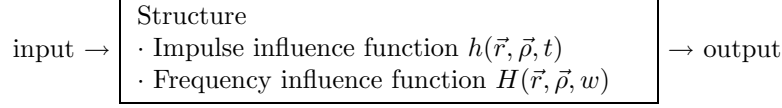
where $\vec{\rho}$ and $\vec{r}$ are position vectors of input and output respectively. $h(\cdot)$ and $H(\cdot)$ are fourier transform pairs.

$$
h(\vec{r}, \vec{\rho}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\vec{r}, \vec{\rho}, w) e^{iwt} dw, \quad H(\vec{r}, \vec{\rho}, t) = \int_{-\infty}^{\infty} h(\vec{r}, \vec{\rho}, t) e^{-iwt} dt
$$
(3.214)

The value of the response functions are zero if the position vectors are not in the domain.

Random response to excitation applied to a point If $F(\vec{\rho}, t)$ is the only input at $\vec{\rho}$. Then $F(x, t) = F(t)\delta(x - a)$.

$$
S_{yy}(\vec{r}, w) = H^*(\vec{r}, \vec{\rho}, w) H(\vec{r}, \vec{\rho}, w) S_{FF}(\vec{\rho}, w)
$$
(3.215)

Response of continuous structures to many direct inputs

$$
S_{ww}(\vec{r}_1, \vec{r}_2, w) = \sum_{j=1}^{N} \sum_{k=1}^{N} H^*(\vec{r}_1, \vec{\rho}_j, w) H(\vec{r}_2, \vec{\rho}_k, w) S_{FF}(\vec{\rho}_j, \vec{\rho}_k, w)
$$
(3.216)

Distributed input to a continuous structure and the corresponding random response

- Given a continuous random input $P(\vec{\rho}, t)$ and a random response $W(\vec{r}, t)$ and initial conditions $W(\vec{r}, 0) = 0$ and $P(\vec{\rho}, t) = 0 \ \forall t < 0$. The response then is:

$$
W(\vec{r}, t) = \int_0^t \int_R h(\vec{r}, \vec{\rho}, t - \tau) P(\vec{p}, \tau) d\vec{\rho} d\tau
$$
(3.217)

- Assuming the inputs are stationary the statistical properties can be calculated as follows:

$$
\begin{aligned}
\mu_w(\vec{r}, t) &= E[w(\vec{r}, t)] = \int_{\infty}^{t} \int_R h(\vec{r}_1, \vec{\rho}, t - \tau) E[P(\vec{p}, \tau)] d\vec{\rho} d\tau \\
R_{ww}(\vec{r}_1, t_1, \vec{r}_2, t_2) &= \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \int_R \int_R h(\vec{r}_1, \vec{\rho}_1, t_1 - \tau_1) h(\vec{r}_2, \vec{\rho}_2, t_2 - \tau_2) \\
& \qquad R_{pp}(\vec{\rho}_1, \vec{\rho}_2, \tau_1 - \tau_2) d\rho_1 d\rho_2 d\tau_1 d\tau_2
\end{aligned}
$$
(3.218)

For convenience define the fourier transform of the correlation as

$$
\Phi_{pp}(\vec{\rho}_1, \vec{\rho}_2, w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{pp}(\vec{\rho}_1, \vec{\rho}_2, u) e^{-iwu} du.
$$
(3.219)

- Recalling the fourier transforms of the cross spectral density of the response, and of the input response function find.

$$S_{ww}(\vec{r}_1, \vec{r}_2, w) = \int_R \int_R \Phi_{pp}(\vec{\rho}_1, \vec{\rho}_2, w) H(\vec{r}_1, \vec{\rho}_2, w) H^*(\vec{r}_2, \rho_2, w) \mathrm{d}\rho_1 \mathrm{d}\rho_2$$
(3.220)

- For practical applications the locations of that the input acts ρ_1, ρ_2 may also be random. Usually $P(\vec{\rho}, t)$ can be assumed to be at least weakly homogenous in space.

$$\Phi_{pp}(\vec{\rho}_1, \vec{\rho}_2, w) = \Phi_{pp}(\vec{\rho}_1 - \vec{\rho}_2) = \Phi_{pp}(\vec{u}, w)$$
(3.221)

and

$$R_{pp}(\vec{\rho}_1, \vec{\rho}_2, \tau) = \Lambda(\rho_1 - \rho_2, \tau) = \Lambda(\vec{u}, w)$$
(3.222)

Where $\vec{u} = \vec{\rho}_1 - \vec{\rho}_2$ and $\tau = t_1 - t_2$.

- The fourier transform can be performed in the space domain *if* the structure is infinite

$$S_{pp}(\vec{u}, w) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi(\vec{k}, w) \mathrm{e}^{i\vec{k}\cdot\vec{u}} \mathrm{d}\vec{k}$$
(3.223)

where $\vec{k}$ is the wave number vector, and $\Psi(\vec{k}, w)$ is the multidimensional wave number spectra. The inverse is:

$$\Psi(\vec{k}, w) = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} S_{pp}(\vec{u}, w) \mathrm{e}^{-i\vec{k}\cdot\vec{u}} \mathrm{d}\vec{u}$$
(3.224)

where α is the number of dimensions. $\Psi(\cdot)$ is sometimes called the wave energy spectra.

- The three dimensional version is most often used. Applications of this theory include 3-D elasticity, acoustics, geophysics and turbulence analysis where the domain can be assumed to be infinite.

$$\vec{\rho} = x\hat{i} + y\hat{j} + z\hat{k}, \quad \vec{k} = k_1\hat{i} + k_2\hat{j} + k_3\hat{k} \quad \text{and} \quad \vec{u} = \xi\hat{i} + \eta\hat{j} + \sigma\hat{k} \quad (3.225)$$

where k_1, k_2 and k_3 are wave numbers and $\xi = x_1 - x_2$, $\eta = y_1 - y_2$ and $\sigma = z_1 - z_2$. The spectral density function of the input then is:

$$S_{pp}(\vec{u}, w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(k_1, k_2, k_3, w) \mathrm{e}^{i(ik_1\xi + k_2\eta + k_3\sigma)} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$$
(3.226)

The inverse can be formulated similarly.

$$\Phi_{pp}(\vec{u}, w) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\xi, \eta, \sigma, w) \mathrm{e}^{-i(ik_1\xi + k_2\eta + k_3\sigma)} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$$
(3.227)

The two dimensional formulation which can be applied to plates or shells is similarly formulated. The one dimensional formulation useful for infinite bars, infinite beams or infinite springs also follows similarly.

Normal Mode Approach to Finite Structures. Assume small damping and orthogonal modes. the general equation of motion is

$$\mathcal{L}[w] + c(\dot{w}) + m_s(w) = P(\vec{\rho}, t) \tag{3.228}$$

here $\mathcal{L}$ is the structural operator. Typical values included $-T\frac{\partial^2}{\partial x^2}$ for a taut string, $EI\frac{\partial^4}{\partial x^4}$ for a beam and $D(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}) + \frac{\partial^4 w}{\partial y^4}$.

The influence response function $h(\vec{r}, \vec{\rho}, t)$ must satisfy the governing equation.

$$\mathcal{L}[h] + c(\dot{h}) + m_s(h) = \delta(\vec{r} - \vec{\rho})\delta t \tag{3.229}$$

Assume a modal solution

$$h(\vec{r}, \vec{\rho}, t) = \sum_{m=1}^{\infty} a_m(\vec{\rho}, t) Y_m(\vec{r}) \tag{3.230}$$

note, each summation is also over the domain. substituting into the equation find:

$$\sum_{m=1}^{\infty} a_m \mathcal{L}[Y_m] + c \sum_{m=1}^{\infty} \dot{a}_m Y_m + m_s \sum_{m=1}^{\infty} \ddot{a}_m Y_m = \delta(\vec{r} - \vec{\rho})\delta t \tag{3.231}$$

First find the value of the normal modes by solving the free vibration solution, notice that the equation can be uncoupled if the nodes are normal so the free modal response of the uncoupled system is of interest.

$$m_2 \ddot{b}_m Y_m + b_m \mathcal{L}[Y_m] = 0 \tag{3.232}$$

For undamped free vibration $\ddot{b}_m = -w_m^2 b_m$. The characteristic condition then is

$$m_s w_m^2 Y_m = \mathcal{L}[Y_m] \tag{3.233}$$

Use orthogonality to find the generalized quantities.

$$
\begin{aligned}
\int_R \cdots \int_R m_s Y_m(\vec{r}) Y_n(\vec{r}) \mathrm{d}r &= \begin{cases} M_n & m = n \\ 0 & m \neq n \end{cases} \\
\int_R \cdots \int_R c Y_m(\vec{r}) Y_n(\vec{r}) \mathrm{d}r &= \begin{cases} c_n & m = n \\ 0 & m \neq n \end{cases}
\end{aligned} \tag{3.234}
$$

Substitute this into the governing equation.

$$M_m \ddot{a}_m(t, \vec{\rho}) + c_m \dot{a}_m(t, \vec{\rho}) + M_m w_m^2 a_m(t, \vec{\rho}) = Y_m(\rho)\delta(t) \tag{3.235}$$

Assume that $a_m = 0 \ \forall t < 0$. The solution for a_m then by separation of variables is

$$a_m(t, \vec{\rho}) = \begin{cases} Y_m(\vec{\rho}) \exp(-\rho_m w_m t) \sin(w_m^\Phi t) & \vec{\rho} \in R \ \& \ t \geq 0 \\ 0 & \vec{\rho} \notin R \text{ or } t < 0 \end{cases} \tag{3.236}$$

where $\rho_m = \frac{c_m}{2 M_m w_m}$ and $w_m^\Phi = w_m \sqrt{1 - \rho_m^2}$. Also $a_m = Y_m(\vec{\rho}) h_m(t) \ \forall \vec{\rho}, t \in \mathcal{D}$. Where $\mathcal{D}$ is the domain where $\vec{\rho} \in R$ and $t \geq 0$. Now the impulse response function reach mode can be written as:

$$h(\vec{r}, \vec{\rho}, t) = \sum Y_m(\vec{r}) Y_m(\vec{\rho}) h_m(t) \quad \forall \vec{\rho}, t \in \mathcal{D} \tag{3.237}$$

The frequency influence function is found by taking the fourier transform

$$H(\vec{r}, \vec{\rho}, w) = \int_{-\infty}^{\infty} h(\vec{r}, \vec{\rho}, t) e^{iwt} \mathrm{d}t = \sum_{m=1} Y_m(\vec{r}) Y_m(\vec{\rho}) H_m(w) \tag{3.238}$$

Where

$$H_m(w) = \frac{1}{M_m(w_m^2 - w^2 + 2i\zeta_m w w_m)} \tag{3.239}$$

Excitation is a stationary random process with zero mean The cross spectral density then is

$$S_{ww}(\vec{r}_1, \vec{r}_2, w) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}_1) Y_n(\vec{r}_2) H_m(w) H_n^*(w) I_{mn}(w) \tag{3.240}$$

where

$$I_{mn}(w) = \int_R \int_r \Phi(\vec{\rho}_1, \vec{\rho}_2, w) Y_m(\vec{\rho}_1) Y_n(\vec{\rho}_2) \mathrm{d}\vec{\rho}_1 \mathrm{d}\vec{\rho}_2 \tag{3.241}$$

The spectral density of the response then occurs when $\vec{r}_1 = \vec{r}_2 = r$

$$S_{ww}(\vec{r}, w) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_m(\vec{r}) Y_n(\vec{r}) H_m(w) H_m^*(w) I_{mn}(w) \tag{3.242}$$

The spectral density is real so $I_{mn} = I_{mn}^*$.

Fatigue

Fractional damage is defined as

$$S_i = \frac{n_i}{N_i} \tag{3.243}$$

where S_i is the stress amplitude n_i are the number of cycles applied and N_i are the number of cycles to failure. **Total Damage** then is the sum

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i} \tag{3.244}$$

when $D \geq 1$ the material fails. This linear damage is known as **Miner's Rule** is a fine rough estimate. The $S - N$ relation is based on a *constant amplitude test*.

Fatigue damage caused by random narrow band stresses (a) Discrete Stress Spectrum. Define n_i as the number of peaks in the stress range ΔS_i. The fraction of stress applied

$$f_i = \frac{n_i}{n} \tag{3.245}$$

where n is the total number of cycles. Also f_i is the probability that a single stress range will have the magnitude S_i. By definition $\sum f_i = 1$. Then the total fatigue damage defined in this term is

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i} = n \sum_{i=1}^{k} \frac{f_i}{N_i}. \tag{3.246}$$

(b) For a continuous stress spectra. For a narrow band process usually assume the Rayleigh distribution of stress ranges. The fraction of stress then in a given range $s \rightarrow s + \Delta s$ is

$$f_i = f_s(s)\Delta s \quad \text{where} \quad f_s(s) = \frac{s}{\sigma_s^2} e^{\frac{-s^2}{2\sigma_s^2}} \tag{3.247}$$

The corresponding damage is

$$D = n \sum_{i=1}^{k} \frac{f_s(s)\Delta s_i}{N(s_i)} \quad \text{as} \quad \Delta s \rightarrow ds \quad \text{thus} \quad D = \int_0^\infty \frac{f_s(s)}{N(s)} ds. \tag{3.248}$$

Using only the linear portion of the $S - N$ curve $NS^m = A$ where A and m are found experimentally. Substituting this into the equation

$$D = \frac{n}{A} \int_0^\infty S^m f_s(s) dS = \frac{n}{A} \int_0^\infty \frac{S^{m+1}}{\sigma_s^2} e^{\frac{-s^2}{2\sigma_s}} ds. \tag{3.249}$$

Solving the integral

$$D = \begin{cases} \frac{V_+\tau}{A}(\sqrt{2}\sigma_s)^m \Gamma(\frac{m}{2} + 1) & \text{where A is based on amplitude} \\ \frac{V_+\tau}{A}(2\sqrt{2}\sigma_s)^m \Gamma(\frac{m}{2} + 1) & \text{where A is based on range.} \end{cases} \tag{3.250}$$

Chapter 4

Mathematical Methods

4.1 Partial Differential Equations

4.1.1 References

- Distribution Theory and Transform Analysis [22]

- Mathematical Methods in Physics and Engineering [5]

- Elementary Applied Partial Differential Equations [10]

- PDE, Prof. Polivani, CU E3102 , Spring 1998

- PDE, Alex Casti, CU E4200, Fall 1999

4.1.2 Glossary

Inner Product:

$$< f, \phi > \quad = \quad < f(t), \phi(t) > \quad \equiv \quad \int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t \qquad (4.1)$$

Function: a rule which relates independent variables to their dependent counterparts.

Testing Function: a guess as to the solution of the function. The space of testing functions D consists of all complex-valued functions $\phi(t)$ that are infinitely smooth and zero outside some finite interval.

Any complex-valued function $f(t)$ that is continuous $\forall\, t$ and zero outside a finite interval can be approximated uniformly by testing functions.

$$|f(t) - \phi(t)| \le \epsilon \quad \text{given } \epsilon > 0 \quad \forall\, t \qquad (4.2)$$

Functional: a rule which assigns a number to each a function in a given set of testing functions.

Distribution: functionals which are *linear* and *continuous*

$$< f, \alpha\phi_1 + \beta\phi_2 > \quad = \quad \alpha < f, \phi_1 > + \beta < f, \phi_2 > \qquad (4.3)$$

$$\lim_{\nu \to \infty} |< f, \phi > - < f, \phi_\nu >| \quad = \quad 0. \qquad (4.4)$$

Local Distribution: Distributions that can be generated from locally integrable functions.

Delta Functional: Not a local distribution.

$$< \delta, \phi > \quad \equiv \quad \phi(0) \qquad (4.5)$$

The shifting property:

$$< \delta(t - \tau), \phi(t) > \quad = \quad < \delta(x), \phi(x + \tau) > \quad = \quad \phi(t). \qquad (4.6)$$

The first derivative:

$$< \delta^{(1)}(t), \phi(t) > \quad \equiv \quad -\phi^{(1)}(0) \qquad (4.7)$$

Convergence: there exists a N_k such that for every $\nu \geq N_k$:

$$|D^k \phi_\nu(t)| \leq \epsilon. \qquad (4.8)$$

Equilibrium or Steady State: Independent of time.

Boundary Conditions: The values of the equation and/or its derivatives at the boundary.

Dirichlet: First kind $\phi = 0$; second kind $\frac{\mathrm{d}\phi}{\mathrm{d}x} = const.$

Principle of Superposition: If u_1 and u_2 satisfy a linear homogenous equation, then an arbitrary linear combination of them $c_1 u_1 + c_2 u_2$ also satisfies the same equation.

Inner product: Again, same as for one dimension only the integral is over the space $\Re^n$.

Sturm-Liouville Problems: Equations of the form

$$\left(py'\right)' - qy + \lambda py = 0 \qquad (4.9)$$

with boundary conditions of the following type

1. $y(a) = 0$ or $y(b) = 0$
2. $y'(a) = 0$ or $y'(b) = 0$
3. $y'(a) - \sigma_1 y(a) = 0$ or $y'(b) - \sigma_1 y(b) = 0$ where $\sigma_1 > 0$, $\sigma_2 > 0$
4. $y(a) = y(b)$ and $p(a)y'(a) = p(b)y'(b)$

5. $y(a)$ and $y'(a)$ are finite and $p(a) = 0$ or $y(b)$ and $y'(b)$ are finite and $p(b) = 0$.

The solutions to the Sturm Liouville Problems form an orthogonal set.

Adjoint Operators:

$$u\left(L(v)\right) - v\left(L^*(u)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\rho\left[u\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\right]\right] \tag{4.10}$$

$$\int_a^b \left[u\left(L(v)\right) - v\left(L^*(u)\right)\right]\mathrm{d}x = \left[\rho\left[u\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\right]\right]_a^b \tag{4.11}$$

If $\left[\rho\left[u\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\right]\right]_a^b = 0$ then L and L^* are adjoint. If $L = L^*$ then the operation L is self adjoint.

$$< u, L(v) >=< v, L^*(u) > \tag{4.12}$$

4.1.3   Solution Approaches

Boundary Value Problems: Separation of Variables ([5]:170–267)

For a Sturm-Liouville type problem with homogenous boundary conditions the solution can be assumed to depend on a product of functions of the separate independent variables. For example: $u(t,\theta) = G(t)\Psi(\theta)$. The resulting odes can be solved and recombined to form the solution.

Distributions: Their Definition and Basic Properties ([22]:1–35)

A distribution is a more appropriate method of representing physical quantities since instantaneous amounts cannot be measured. Any uniform function can be represented by a series of testing functions.

Testing Functions and Distributions can be applied to systems of Several Variables. For a testing function of several variables t_i differentiated to order $\hat{k} = \{k_1, k_2, \ldots, k_i, \ldots, k_n\}$.

$$D^k\phi(t) \equiv \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial t_1^{k_1}\partial t_2^{k_2}\cdots t_n^{k_n}}\phi(t_1, t_2, \ldots, t_n) \tag{4.13}$$

Two distributions f and g are equal if

$$< f, \phi > \quad = \quad < g, \phi > \tag{4.14}$$

for every testing function $\phi(t)$ in D.

Some Operations of Distributions:

Addition: $< f + g, \phi > \ \equiv \ < f, \phi > + < g, \phi >$

Multiplication by Constant: $< \alpha f, \phi > \ \equiv \ < f, \alpha \phi >$

Shifting: $< f(t - \tau), \phi(t) > \ \equiv \ < f(t), \phi(t + \tau) >$

Transposition: $< f(-t), \phi(t) > \ \equiv \ < f(t), \phi(-t) >$

Multiplication of independent Variable by a Constant:

$$< f(\alpha t), \phi(t) > \equiv \left\langle f(t), \frac{1}{a^n} \phi\left(\frac{t}{a}\right) \right\rangle \qquad (4.15)$$

Multiplication by an Analytic Function:

$$\langle fg, \phi \rangle \ \equiv \ \int_{\Re^n} f(t) g(t) \phi(t) \mathrm{d}t \qquad (4.16)$$

Green's Functions   ([10]:370–433, 495–521)

Given $L(u) = f$ then $u = \int G(x, x_0) f(x_0) \mathrm{d}x_0$ where $LG = \delta(x - x_0)$.

Use eigenfunction expansion to prove self adjointness of the Laplacian.

$$
\begin{aligned}
\int_R (v \nabla^2 u + u \nabla^2 v) \mathrm{d}x \mathrm{d}y \ &= \ \oint (u(\nabla v) - v(\nabla u)) \cdot \vec{n} \mathrm{d}S \\
&= \ \int_R (-\phi_1 \lambda_2^2 \phi_2 + \phi_2 \lambda_1^2 \phi_1) \mathrm{d}x \mathrm{d}y \\
&= \ (\lambda_1^2 - \lambda_2^2) \int \phi_1 \phi_2 \mathrm{dxdy} = 0 \qquad (4.17)
\end{aligned}
$$

The eigenfunctions are orthogonal so the equality is true.

4.1.4   Examples of Linear Partial Differential Equations

Heat Equation   ([10]:1–18)

Heat energy is conserved over an arbitrary subregion: rate of change of eat energy = heat energy flowing across the boundaries per unit time + heat energy generated inside per unit time.

$$c(\vec{x}) \rho(\vec{x}) \frac{\partial \vec{u}}{\partial t} = \nabla \cdot (K_0(\vec{x}) \ \nabla \vec{u}) + Q \qquad (4.18)$$

ρ: density
c: specific heat
Q: Heat Source
K_0: Proportionality constant between heat flux and the change in temperature

per unit length, thermal conductivity. From **Fourier's law of Heat Conduction**: $\phi(\vec{x}) = -K_0(\vec{x})\nabla u$.

Boundary conditions and initial conditions need to be satisfied to solve the problem, homogenous boundary conditions are amenable to separation of variables.

Vibrating Strings and Membranes ([10]:130–149)

Using conservation of linear momentum for perfectly elastic springs:

$$\rho_0(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = T_0 \nabla^2 \vec{u} + Q(\vec{x}, t)\rho(\vec{x}) \tag{4.19}$$

D'Alembert's solution is in the form $u(x, t) = R(x - ct) + S(x + ct)$ where R and S are some functions which are differentiable and satisfy the boundary conditions.

Laplace Transform ([13]:525–529,[10]:568–598)

The Laplace transform of the function $f(t)$ is:

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{4.20}$$

Use the Heaviside unit step function $\left(H(t - b) = \left\{ \begin{array}{ll} 0 & t < b \\ 1 & t > b \end{array} \right\} \right)$ to force function $f(t)$ to be 0 when $t < 0$.

The derivatives of this distribution are then used to solve the partial differential equations:

$$L[f'(t)] = sF(s) - f(0) \tag{4.21}$$

$$L[f''(t)] = s^2 F[s] - sf(0) - f'(0) \tag{4.22}$$

Solve the resulting transformed PDE for $F(s)$ and use the inverse transform to find $f(t)$.

$$L^{-1}[F(s)] = f(t) \tag{4.23}$$

4.2   Numerical Methods

4.2.1   References

- Applied Numerical Analysis [9]

- Complexity, Prof. Henrik Wozinkowski, CU W4241, Fall 1999

- Numerical Methods, Prof. Wright, CU E4300, Spring 2000

4.2.2 Glossary

Complexity: the minimum time needed to solve a problem. If no discrete complexity is known find lower and upper bounds on computing time. Assume real number model when calculating complexity. Consequently, complexity does not depend on the algorithm or size of numbers. It is controlled only by the computational difficulty of the problem and the set of permissible operations.

Algebraic Complexity: the minimum time needed to solve a problem when complete information is given and exact solutions can be found.

Real Number Model: Arithmetic operations can be performed exactly with unit cost. No matter how large the numbers being operated upon are the cost of the operation remains the same. The assumption is valid since most modern computers use floating point arithmetic.

Floating Point Arithmetic: Every number is represented by $x = m2^c$ where $\frac{1}{2} \le \|m\| < 1$ and $c \in Integer$. The error in the floating point operation is related to the mantissa bit t.

$$m \approx \sum_{i=1}^{t} b_i 2^{-t} \quad b_i \in [0,1]$$

If $t = \infty$ no rounding errors would exists. But t is finite and consequently $fl(x) = x(1 - \epsilon x)$ where $\|\epsilon x\| = 2^{-t}$.

Numerical Stability: the computed result is the exact solution for a slightly perturbed data.

Roundoff Error: Is related to how a computer holds numbers and is generally on the order of 10^{-12}. It is generated by operations, addition, multiplication etc.

Singular Matrix: If a $n \times n$ matrix $[A]$ is singular it does not produce a unique solution for set of linear equations $[A]\{x\} = \{b\}$.

Gauss Elimination: Use division and addition of systems of equations to put A in the form of a lower triangular matrix and solve $Ax = b$ by back substitution. The linear system is solved in n^3.

LU Factorization: Any process which converts A to LU. Gauss Elimination is an example LU factorization.

Eigenvalue, Eigenvector Properties: $A\vec{x} = \lambda\vec{x} \quad \forall \vec{x} \ne 0 \quad \exists \quad \lambda$ which satisfies the equation. λ can be found by solving the singular matrix

$$\det |A - \lambda \mathtt{I}| = 0. \tag{4.24}$$

Error: The error at step m is defined as $\left|e^{(m)}\right| = \left|x^{(m)} - r\right|$ where r is the actual value and $x^{(m)}$ is the value at that iterative step. Note, $r, x \in \Re^n$

Convergence: An iteration converges if the error is smaller each subsequent step.

Asymptotic Order of Convergence:

$$\lim_{m \to \infty} \frac{\left|e^{(m+1)}\right|}{\left|e^{(m)}\right|^p} = K \tag{4.25}$$

For convergence to occur $K < 1$, p is the order of convergence.

Consistent: A finite difference approximation that matches the taylor series up to order p is called consistent to $O(h^p)$.

Fundamental Theorem of Numerical Stability: A necessary and sufficient condition for convergence to the correct solution is (a) consistency and (b) stability.

Amplification Factor: A test of stability for ordinary differential equations is solving the equation for the simplest function $y' = Ay$ where the solution is known $y(x) = e^{Ax}$. A solved using the method is the amplification factor.

4.2.3   Solving Sets of Equations $[A]\{x\} = [B]$

Direct Method

The most common direct method is Gaussian Elimination it is an order n^3 operation. LU decomposition and others are versions of Gaussian Elimination.

Fixed-Point Iteration in n Dimensions:

The **basic theorem** for fixed point iteration is If $|g'(x)| < 1 \quad \forall x \in [a, b]$ surrounding a fixed point r, defined by $r = g(r)$, then iteration of the form $x = g(x)$ converges to the fixed point r for any starting point in the interval $[a, b]$. The **generalized** form $\vec{X} = \vec{G}(\vec{X})$ where $X \in \Re^n$ and $\vec{G} \in \Re^n$ converges to the fixed point R for any starting point F for which the spectral radius of the iteration matrix is strictly less than one

$$\rho\left(\vec{G}(\vec{X})\right) < 1. \tag{4.26}$$

Newton Iteration is used to solve the root finding equation $\vec{F}(\vec{X}) = 0$ the following fixed point iteration equation

$$\vec{G}(\vec{X}) = \vec{X} - [\vec{F}'(\vec{X})]^{-1}\vec{F}(\vec{X}). \tag{4.27}$$

Jacobi Iteration is used to solve the system $A\vec{x} = \vec{b}$ by separating the A matrix into $A = L + D + U$ where L is a lower triangular matrix D is a diagonal and U is an upper triangular matrix. The system is solved by fixed point iteration $\vec{x} = b' - G'\vec{x}$ where $G' = D^{-1}(L+U)$. **Gauss Seidel Iteration** is used to solve the system $A\vec{x} = \vec{b}$ by separating the A matrix into $A = L+D+U$ where L is a lower diagonal matrix D is a diagonal and U is an upper diagonal matrix. The system is solved by fixed point iteration $\vec{x} = b' - G'\vec{x}$ where $G' = (L+D)^{-1}U$.

Eigenvalue Problem:

Use the **Power Method** to solve $Ax = \lambda x$ guess an x_o solve for $Ax_0 = z_1$ normalize z_1 by dividing by its largest value member v_1 and repeat $Av_1 = z_2$. A **Shifted Matrix** has the matrix $(A - s\mathtt{I})$ has the same eigenvector as A and its eigenvalue is $\lambda - s$.

Problem	Methods	Comments on p, K
f(x) = 0 (scalar)	Bisection	$p = 1$, $K = 1/2$
	Secant	$p = 1.618...$, K finite
	Newton	$p = 2$, K finite
	Fixed-point iteration of form $x = g(x)$	$p = 1$, $K = \|g'(x)\| < 1$
$Ax = \lambda x$	Power method with A; find dominant eigenvalue λ_1 of A.	$p = 1$, $K = \max_{i \neq 1} \frac{\|\lambda_i\|}{\|\lambda_1\|}$
	Power with $(A - s\mathtt{I})^{-1}$; find λ_i, closest to s,	$p = 1$, $K = \max_{i \neq j} \frac{\|\lambda_i - s\|}{\|\lambda_j - s\|}$
Ax =b where $A = L + D + U$	Fixed-point iteration of the form $x = c - Bx$ Jacobi: $c = D^{-1}b$, $B = D^{-1}(L + U)$ Gauss Seidel: $c = (L + D)^{-1}b$, $B = (L + D)^{-1}U$	$p = 1$, $K = \max_i \|\lambda_i\| < 1$ where λ_i are the eigenvalues of B.
F(X) = 0 ($\in \Re^{\mathtt{n \ x \ n}}$)	Fixed-point iteration of the form $X = G(X)$ For Newton's method $G(X) = X - [F'(X)]^{-1}F(X)$, where $[F'(X)]$ is the Jacobian matrix of $F(X)$	$p = 1$, $K = \max_i \|\lambda_i\| < 1$ where λ_i are the eigenvalues of $[G'(X)]$, the Jacobian matrix of $G(X)$ p =2, K finite

Figure 4.1: Iterative Methods

4.2.4   Numerical Calculus

Finite Difference Approximations

The taylor expansion of a series around a point x_0 is

$$f(x_0 + h) = f(x_) + hf'(x_0) + \frac{hf''(x_0)}{2!} + \ldots + h^n \frac{f^{(n)}(x_0)}{n!} + \ldots \qquad (4.28)$$

The finite difference approximation for the first derivative f' is:

$$f'(x_0) = \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \approx \frac{f(x_0+h)-f(x_0)}{h} \tag{4.29}$$

Using the taylor series expansion it can be shown that the truncation error from this approximation is $O(h)$. The roundoff error is also a factor since the method requires a subtraction of nearly equal numbers. Higher order difference equations are found by combining the taylor expansions at different points, or converging from a different direction.

For example **Richardson's Extrapolation** is formed from the following two expansions:

$$f(x_0+h) = f_0+hf_0'+\frac{1}{2}h^2f_0''+O(h^3) \quad \text{and} \quad f(x_0-h) = f_0-hf_0'+\frac{1}{2}h^2f_0''-O(h^3). \tag{4.30}$$

For $f_0 = f(x_0)$, $f_1 = f(x_0+h)$, and $f_{-1} = f(x_0-h)$:

$$f_0' = \frac{f_1-f_{-1}}{2h}+O(h^2)+O(h^4)\cdots \tag{4.31}$$

Repeat process and find f_0' at two different points to find an $O(h^4)$ truncation error.

Method of Undetermined Coefficients

For *derivation*: given a set of points x_i find an approximation formula of the form $\sum_i^{n+1} \vec{c}\cdot\vec{f}$ where the basis of $\vec{f}$ is a set of functions. If the formula $\hat{\mathbf{f}}$ is expanded using the set of power law expansion functions $\vec{f} = \{1, x, x^2, \ldots, x^n\}^T$. The approximation is exact up to a truncation error of $O(h^{n+1})$. Solve the system of $n \times n$ equations:

$$\begin{array}{ccccccccc}
f_0' & = & c_0 f_0 & + & c_1 f_1 & + & c_2 f_2 & \cdots & c_n f_n \\
f_0'' & = & c_0 f_0' & + & c_1 f_1' & + & c_2 f_2' & \cdots & c_n f_n' \\
\vdots & & \vdots & & \vdots & & \vdots & \ddots & \vdots \\
f_0^{(n)} & = & c_0 f_0^{(n)} & + & c_1 f_1^{(n)} & + & c_2 f_2^{(n)} & \cdots & c_n f_m
\end{array} \tag{4.32}$$

For *integration*: again assume that the solution is a linear combination of the test functions

$$\int_a^b f(x)\mathrm{d}x = \sum_{i=1}^n c_i f_i \tag{4.33}$$

For example the **Trapezoidal rule** can be found using two points in the expansion $\int_a^b f(x)\mathrm{d}x = c_0 f(a) + c_1 f(b)$ apply the method.

$$\int_a^b (1)\mathrm{d}x = c_0(1) + c_1(1) = b - a$$
$$\int_a^b x\mathrm{d}x = c_0(a) + c_1(b) = \frac{b^2}{2} - \frac{a^2}{2} \tag{4.34}$$

Thus $c_0 = (b-a)/2$ and $c_1 = (b-a)/2$ and

$$\int_a^b f(x) \approx \frac{b-a}{2}[f(a) + f(b)]. \tag{4.35}$$

To find **Simpson's rule** take three points $\int_a^b f(x)\mathrm{d}x = c_0 f(a) + c_1 f\left(\frac{a+b}{2}\right) + c_2 f(b)$.

$$
\begin{aligned}
\int_a^b \mathrm{d}x &= c_0 + c_1 + c_2 = b - a \\
\int_a^b x\mathrm{d}x &= c_0 a + c_1 \left(\frac{a+b}{2}\right) + c_2 b = \frac{b^2}{2} - \frac{a^2}{2} \\
\int_a^b x^2\mathrm{d}x &= c_0 a^2 + c_1 \left(\frac{a+b}{2}\right)^2 + c_2 b^2 = \frac{b^3}{3} - \frac{a^3}{3}
\end{aligned} \tag{4.36}
$$

Solve to find $c_0 = c_2 = \frac{1}{6}(b-a)$, $c_1 = \frac{4}{6}(b-a)$. Hence

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{4.37}$$

More commonly $b - a = 2h$ and $a = x_0$

$$\int_{x_0}^{x_0+2h} f(x)\mathrm{d}x \approx \frac{h}{3}(f_0 + 4f_1 + f_2). \tag{4.38}$$

The extended trapezoidal and expanded simpson's rules are sums over n divisions of the cross section.

Gaussian Quadrature

Similar to the method of undetermined coefficients, if the bounds on the integral is known then the points at which the values are taken inside the bounds can be derived.

$$\int_{-1}^1 f(t) \approx af(t_1) + bf(t_2) \tag{4.39}$$

This equation must be true for a polynomial of degree three, hence

$$
\begin{aligned}
f(t) = 1; &\quad \int_{-1}^1 \mathrm{d}t = 2 = a + b \\
f(t) = t; &\quad \int_{-1}^1 t\mathrm{d}t = 0 = at_1 + bt_2 \\
f(t) = t^2; &\quad \int_{-1}^1 t^2\mathrm{d}t = \frac{2}{3} = at_1^2 + bt_2^2 \\
f(t) = t^3; &\quad \int_{-1}^1 t^3\mathrm{d}t = 0 = at_1^3 + bt_2^3
\end{aligned} \tag{4.40}
$$

Find the governing equation $0 = 0 + b(t_2^3 - t_2 t_1^2) = b(t_2)(t2 - t_1)(t_2 + t_1)$. The only equality which can allow a term to go to zero and satisfy the equation without reducing the formula to one term is $t_2 = -t_1$. Thus $a = b = 1$ and $t_2 = -t_1 = \sqrt{\frac{1}{3}}$. Approximations with more points require more terms.

Ordinary Differential Equations

As for the finite difference approximation include a desired number of terms of the taylor series expansion and an error term:

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{y''(\zeta)h^2}{2} \qquad x_0 < \zeta < x_0 + h \qquad (4.41)$$

Euler's Method is written in the form

$$y_{n+1} = y_n + hy'_n \qquad (4.42)$$

Equations to be solved are in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$. Start from the initial conditions, choose an h and continue evaluating until the iteration converges. Thus the values of x are $x_o, x_0 + h, x_0 + 2h$ and so on.

The **Runge Kutta Method** is equivalent to matching the first n terms of the taylor expansion, but can be written in compacted form. The second order method is equivalent to a modified euler method for $a + b = 1$, $\alpha b = \frac{1}{2}$, and $\beta b = \frac{1}{2}$.

$$y_{n+1} = y_n + ak_1 + bk_2 \quad \text{where} \quad k_1 = hf(x_n, y_n) \text{ and } k_2 = hf(x_n + \alpha h, y_n + \beta k_1) \qquad (4.43)$$

The more commonly used fourth order Runge Kutta method is:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= hf(x_n, y_n) \\
k_2 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right) \\
k_4 &= hf(x_n + h, y_n + k_3) \qquad (4.44)
\end{aligned}$$

Boundary-Value Problems

Shooting Method Suppose a value $f(x)$ at $x = a$ and $x = b$ is known. If $f'(a)$ were known this would be an initial value problem and solved accordingly. Solve iteratively to find $f(b)$ by guessing $f'(a)$.

Finite Difference can be used for linear equations. The resulting equation matrix can be solved directly by gaussian elimination or iteratively. Or as a **Characteristic Value Problem** where the matrix is singular and the determinant must also be zero for a unique solution to exist.

4.3   Finite Element Method

4.3.1   References

- Finite Element Method

- Finite Element, Prof. Raimondo Betti, CU E4332, Fall 1999

4.3.2   Glossary

Interpolating Function: a power series or other polynomial expansion of a continuous Function.

Shape function: A function which is given independently for each individual node of a finite element. It returns a value of one at the node and zero at every other node inside the element.

Stiffness Matrix: Generally it is the correspondence between input and output. In specific terms, if $\{u\}$ is the displacement vector and $\{f\}$ is the vector of forces applied at each node then:

$$\{f\} = [k]\{u\} \tag{4.45}$$

where $[k]$ is the stiffness matrix.

Connectivity: is the description of how elements are connected.

Rayleigh-Ritz Method: guess that the displacement can be given in the following form.

$$\widetilde{u} = a_1 f_1(x) + a_2 f_2(x) \ldots = a_i f_i(x) \tag{4.46}$$

The $u(x)$ must be chosen such that the boundary conditions are satisfied and the individual $f_i(x)$ are linearly independent. This guess is substituted into the functional:

$$I(a_1, a_2, \ldots) = \int_a^b \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} \sum c_i v_i \right)^2 - Q \left(\sum c_i v_i \right)^2 + 2F \sum c_i v_i \right] \mathrm{d}x \tag{4.47}$$

This functional corresponds with the second-order linear boundary value problem over $[a, b]$ with Dirlechet boundary conditions $y'' + Q(x)y = F(x)$ and $y(a) = y_0$ and $y(b) = y_n$. Neuman boundary conditions result in a different functional.

To find the minimum of the functional take derivatives in terms of a_i.

$$\frac{\partial I}{\partial a_i} = \int_a^b 2 \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \frac{\partial}{\partial c_i} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \mathrm{d}x - \int_a^b 2Qu \left(\frac{\partial u}{\partial c_i} \right) \mathrm{d}x + 2 \int_a^b F \left(\frac{\partial u}{\partial c_i} \right) \mathrm{d}x \tag{4.48}$$

Collocation Method: The residual of given ordinary differential equation $y'' + Q(x)y = F(x)$ is $R(x) = y'' + Qy = F$. Again approximate $y(x)$ with a system of trial functions $u(x)$. Substitute $u(x)$ into the equation and attempt to make $R(x) = 0$

Galerkin Method: Instead of attempting to find the residual function $R(x)$ at zero the function times a weighting function is evaluated:

$$\int_a^b W_i(x)R(x)\mathrm{d}x = 0, \quad i = 0, 1, \ldots, n \tag{4.49}$$

Variational Formulation: minimize the energy function, the minimization returns the governing equation if the function is known and an approximation to it if the function is approximated.

Strain Energy Density:

$$\mathrm{d}W = \sigma_x \mathrm{d}\epsilon_x + \sigma_y \mathrm{d}\epsilon_y + \sigma_z \mathrm{d}\epsilon_z + \tau_{xy}\mathrm{d}\gamma_{xy} + \tau_{yz}\mathrm{d}\gamma_{yz} + \tau_{zx}\mathrm{d}\gamma_{zx} \tag{4.50}$$

$$\mathrm{d}W = [\sigma]\mathrm{d}\epsilon^T, \quad \mathrm{d}W = [E]\epsilon \ \mathrm{d}\epsilon^T, \quad W = \int_0^\epsilon [E]\epsilon \mathrm{d}\epsilon^T = \frac{1}{2}\epsilon^t[E]\epsilon \tag{4.51}$$

Conservative Force: path the work is done in is irrelevant.

Consistent Nodal Forces: Statically equivalent to actual forces.

Essential Boundary Condition: usually a geometric boundary condition.

Natural Boundary Condition: usually a displacement boundary condition.

Dirichlet Boundary Condition: Values at the boundary are constant.

Neuman Boundary condition: a constant times the derivative at the boundary is constant (for example, flux or stress from a linear stress strain equation).

Lagrangian Interpolation: A linear element is defined by 2 nodes, higher order elements must be defined by order $+1$ nodes.

The lagrangian polynomial can be used for interpolation, 1 at the node 0 elsewhere:

$$N_i(x) = \frac{(x_1 - x)(x_2 - x)\cdots(x_{i-1} - x)(x_{i+1} - x)\cdots(x_n - x)}{(x_1 - x_i)(x_2 - x_i)\cdots(x_{i-1} - x_i)(x_{i+1} - x_i)(x_{i+1} - x)\cdots(x_n - x_i)} \tag{4.52}$$

Isoparametric Element: An isoparametric element is in the form $\phi = a_1 + a_2x + a_3y + a_4z$. It is continuous only up to C^0. For it the shape functions for mapping and displacement are of the same type, order and value. Generally well behaved for numerical integration.

Superparametric Element: Mapping to quadrature plane of a higher order than the shape functions of the displacement functions.

Subparametric Element: Mapping to quadrature plane of lower order mapping than displacement function.

Function Class: C^n only n derivatives are allowed, after n the derivatives are discontinuous.

4.3.3 Theory

Finite Element Method for Ordinary Differential Equations

- Subdivide [a,b] into n subintervals, or elements, that join $x_1, x_2, \ldots, x_{n-1}$. Add to this array the boundary conditions $x_0 = a$ and $x_n = b$. Now x_i are nodes of the interval, and are not necessarily evenly spaced.

- Apply Galerkin method to each element separately to interpolate (subject to a given differential equation) between $u(x_{i-1})$ and $u(x_i)$. Where the u's are approximations to the $y(x_i)'s$ that are the true solution to the differential equations. The nodal values become the a_i.

- Use the lowest -degree polynomial for $u(x)$.

- The result of applying Galerkin to element i is a pair of equations which have the unknowns as the nodal values at the ends of the element, or the a_i. Combining the unknowns for each element gives a system of linear equations.

- apply boundary conditions, get intermediate values for y(x) by linear interpolation.

Finite Element Method for Elliptic Partial Differential Equations

A general elliptic equation is $u_{xx} + u_{yy} + Q(x,y)u = F(x,y)$ on a region R that is bounded by a curve L with the boundary conditions $u(x,y) = u_0$ on L_1 and $\frac{\partial u}{\partial n} = \alpha u + \beta$ on L_2 where $\frac{\partial u}{\partial n}$ is the unit outward normal gradient.

- Find the functional that corresponds to the partial-differential equation. This functional can be derived from the theory of **Virtual Work** and is the **Principle of Stationary Potential Energy** if the system is at rest.

- Subdivide into subregions or elements. Every node and every side of the elements must be common with adjacent elements except for sides on the boundaries.

- Write an interpolating relation that gives values for the dependent variable within an element based on the values of the nodes. The interpolation should be a sum of n products of a_i and a function $v_i(x,y)$ for an n noded element.

- Substitute the interpolating relation into the functional and set the partial derivatives with respect to each node a_i to zero.

- Combine together the elements and solve the resulting system of equations after the boundary conditions have been imposed. The value of $u(x, y)$ in the interior region is given using the interpolating relations.

The Governing Functional

Strain energy density is related to body force $-f_i = \frac{\partial w_f}{\partial u_i}$ and surface force as follows $-t_i = \frac{\partial w_t}{\partial u_i}$. The potential energy function [1]:

$$\Pi_p = \int_V w(\widetilde{u}) \mathrm{d}V - \int_V f_i u_i \mathrm{d}V - \int_A t_i u_i \mathrm{d}A - \sum P_i u_i \qquad (4.55)$$

For axial loading and axial displacement, using the strain energy density formulation:

$$\Pi_p = \frac{1}{2} \int_0^L \epsilon_x [E] \epsilon_x A \mathrm{d}x - \int_0^L u q(x) dx \qquad (4.56)$$

Numerical Integration– Gaussian quadrature

$$\int_{-1}^1 f(\zeta) \mathrm{d}\zeta = \sum_{i=1}^N f(\zeta_i) W_i + R \approx \sum_{i=1}^N f(\zeta_i) W_i \qquad (4.57)$$

For a polynomial of points $\zeta_1, \zeta_2, \ldots$ the solution is exact if the points are chosen as gauss points.

Since the numerical integration is from $\{-1, 1\}$ each element must be mapped into a square symmetrical plane. The Jacobian is used for this $\mathrm{d}x\mathrm{d}y = |J|\mathrm{d}\zeta\mathrm{d}\eta$.

Two Dimensional Elements

Two dimensional elements with n nodes are in general represented by

$$\phi(x, y) = N_1(x, y)\phi_1 + N_2(x, y)\phi_2 + \ldots + N_n(x, y)\phi_n \qquad (4.58)$$

---

[1]

Stationary Potential Energy Assumptions: (1) Conservative forces, (2) no discontinuity (3) essential boundary conditions are satisfied.

The Potential Energy Function: Π_p

$$\Pi_p = \int w \mathrm{d}V + \int w_f \mathrm{d}V + \int w_t \mathrm{d}A + \Pi_{pc} \qquad (4.53)$$

Energy is a scalar value and the energy contribution of each part of the system can be added linearly due to the conservative function assumption.

$$\Pi_{pi} = \frac{1}{2} \int_{v_1} \epsilon^T [E] \epsilon \, \mathrm{d}V \qquad (4.54)$$

Where N_i are the shape functions and ϕ_i are the nodal values. Use Linear Lagrangian interpolation, first along the vertical nodes and then along the horizontal.

Equilibrium and Compatibility:

- **At the nodes:** equilibrium of forces and compatibility are always satisfied.

- **At the boundaries:** equilibrium *not* usually satisfied. more elements lead to less stress and unbalance in the boundary. In the limit equilibrium is satisfied.

- **Inside elements:** equilibrium is usually not satisfied.

4.3.4   Application

Bar

Governing Equation: $\vec{p} = [K]\vec{u}$ where $\vec{p}$ is the force applied at each node, $[K]$ is the stiffness matrix and $\vec{u}$ is the displacement at each node.

One linear elastic one-dimensional bar in its local coordinate system:

$$\{\bar{p}\} = \begin{Bmatrix} \bar{P}_{x_i} \\ \bar{P}_{y_i} \\ \bar{P}_{x_j} \\ \bar{P}_{y_j} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{u}_j \\ \bar{v}_j \end{Bmatrix} = [\bar{K}]\{\bar{u}\}. \tag{4.59}$$

Using the stationary potential energy formulation:

$$\Pi_p = \frac{1}{2} \int \epsilon_x E \epsilon_x A \mathrm{d}x - \int_0^L u q(x) \mathrm{d}x \tag{4.60}$$

Finite Element Approach

Rewrite the principle of stationary potential energy in terms of displacements. Energy is a scalar so the energies of the components can be added together without regard to direction.

$$\Pi_{p_i} = \frac{1}{2} \int_{V_i} \widetilde{\epsilon}^T [E] \widetilde{\epsilon} \mathrm{d}V - \int_{V_i} \widetilde{u}^T f \mathrm{d}V - \int_A \widetilde{u}^T \widetilde{t} \mathrm{d}A - \Pi_c^i \tag{4.61}$$

1. Find expression of ϵ in terms of u. Using a linear approximation $u = m\bar{x} + b = \frac{u_j - u_i}{L} + u_i$. Parametrize to find the value of u in between the nodes, and according to the interpolation: $u = \frac{L-\bar{x}}{L}u_i + \frac{\bar{x}}{L}u_j = [N]\{\vec{u}\}$.

$$\epsilon_x = \frac{\mathrm{d}u}{\mathrm{d}\bar{x}} = [\, -\tfrac{1}{L} \quad \tfrac{1}{L} \,] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{4.62}$$

Build the strain energy matrix.

$$\frac{1}{2}\widetilde{\epsilon}^T[E]\epsilon = \frac{1}{2}\{\, u_i \quad u_j \,\} \frac{EA}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{4.63}$$

The general stiffness matrix expression: $[B]^T[E][B]$ where $[B]$ is the gradient of the shape function $[N]$.

2. Find applied forces.

$$\int_V \bar{u}^T f \mathrm{d}V = \{\, u_i \quad u_j \,\} \left\{ \begin{array}{c} \int_0^L N_1(\bar{x}) f \mathrm{d}\bar{x} \\ \int_0^L N_2(\bar{x}) f \mathrm{d}\bar{x} \end{array} \right\} = -\{\, u_i \quad u_j \,\} \left\{ \begin{array}{c} r_1^f \\ r_2^f \end{array} \right\} \quad (4.64)$$

The $1 - D$ potential energy formulation.

$$\begin{aligned}
\Pi_p \;=\; & \frac{1}{2}\{u_i\}^T\{u_i\}\frac{EA}{L} \\
& + \frac{1}{2}\{\, u_1 \quad u_2 \,\}[K_{12}]\{\begin{array}{c} u_1 \\ u_2 \end{array}\} + \frac{1}{2}\{\, u_2 \quad u_3 \,\}[K_{23}]\{\begin{array}{c} u_2 \\ u_3 \end{array}\} \cdots \\
& - \{u_1\}\{r_1^f + r_1^t + r_1^c\} - \{\, u_2 \quad u_3 \,\}\{\begin{array}{c} \dot{r}_1^f + \dot{r}_1^t + \dot{r}_1^c \\ \dot{r}_2^f + \dot{r}_2^t + \dot{r}_2^c \end{array}\} \cdots \quad (4.65)
\end{aligned}$$

3. Use principal of potential energy to solve for the equilibrium state, using essential boundary conditions.

$$\frac{\partial \Pi_p}{\partial u_i} = 0, \quad [K]\{u_i\} = \{R_i\} \quad (4.66)$$

Truss

A truss can be represented as a combination of one dimensional bars. The angle of the bar coordinates to the global coordinate system is θ. The final matrix equation is a linear superposition of each component bar: $\{p\} = [T]^T[\bar{K}][T]\{u\}$, where $[T]$ is the rotation matrix.

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad (4.67)$$

It may be convenient to segment the resulting matrix into known an unknown parts. According to the uniqueness requirements of the continuum equations either forces or displacements need to be prescribed at every node.

To find displacements of a **truss** combine the matrix into the form $[K]\{u\} = \{P\}$ impose boundary conditions – either forces or displacements — to solve. Since stresses are constant along each axial member the internal forces are also known.

The **penalty method** is used to account for a null column in the stiffness matrix.

Beam

Governing Equation: $\kappa = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{M}{EI}$, in a two dimensional isotropic beam in which Poisson's effect has been neglected and the plane sections are assumed to be planar. κ is the curvature and EI is the bending stiffness.

$$S = \frac{\mathrm{d}M}{\mathrm{d}x}, \quad q = \frac{\mathrm{d}S}{\mathrm{d}x} = \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = EI\frac{\mathrm{d}^3 \theta}{\mathrm{d}x^3} \tag{4.68}$$

Integrate to find the constants for each of the beam elements. Recombine to create a matrix of the following form $\{\widetilde{P}\} = [K]\{\widetilde{u}\}$:

$$\begin{Bmatrix} \bar{P}_i \\ \bar{M}_i \\ \bar{P}_j \\ \bar{M}_j \end{Bmatrix} = EI \begin{pmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{6}{L^3} \\ \frac{-6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{pmatrix} \begin{Bmatrix} \bar{v}_i \\ \theta_i \\ \bar{v}_j \\ \ddot{\theta}_j \end{Bmatrix} \tag{4.69}$$

Either the forces at the boundary or the displacements need to be known to solve.

Shape Functions For a Rectangle: The behavior at any point in the rectangle as defined by the nodal points is approximated by:

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 \tag{4.70}$$

In general the interpolation function for a rectangle is

$$\phi = A + Bx + Cy + Dxy \tag{4.71}$$

Now the value of ϕ is known at the nodal points for $\phi(0,0) = \phi_1$, $\phi(a,0) = \phi_2$, $\phi(a,b) = \phi_3$ and $\phi(0,b) = \phi_4$.

$$\phi_1 = A \quad \phi_2 = A + Ba \quad \phi_3 = A + Ba + Cb + Dab \quad \phi_4 = A + Cb \tag{4.72}$$

Substituting into the interpolation function find:

$$N_1 = \frac{(a-x)(b-y)}{ab} \quad N_2 = \frac{x(b-y)}{ab} \quad N_3 = \frac{xy}{ab} \quad N_4 = \frac{y(a-x)}{ab} \tag{4.73}$$

Isoparametric Elements

Isoparametric elements are two dimensional finite elements in a ζ, η plane with four nodes, find the interpolation function if the origin is at the center of the square and the lenghts of the sides are two:

$$\Phi(\xi, \eta) = N_1\Phi_1 + N_2\Phi_2 + N_3\Phi_3 + N_4\Phi_4 \tag{4.74}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \tag{4.75}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta) \tag{4.76}$$

The geometrical coordinates $\vec{x}$ can now be written in terms of this shape function, here k is the number of nodes:

$$\vec{x}(\vec{\xi}) = \sum_{i=1}^{k} N_i \vec{x}_i \tag{4.77}$$

where $\vec{x}_i$ is the location of the coordinate i in geometrical coordinates. All other properties pertaining to the problem can be distributed similarly (in cartesian coordinates).

$$\vec{u}(\xi, \eta) = \sum_{i=1}^{k} N_i \vec{u}_i \quad \text{and strain} \quad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j} + \frac{\partial u_j}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} \right) \tag{4.78}$$

The Jacobian matrix is

$$J = \frac{\partial \vec{x}}{\partial \vec{\xi}}. \tag{4.79}$$

The stiffness matrix then is

$$[K] = \int_{-1}^{1} \int_{-1}^{1} t[B]^T [E][B] |J^{-1}| \mathrm{d}\xi \mathrm{d}\eta \tag{4.80}$$

Here t is the thickness and $[B] = J^{-1}[N_{,\xi}]$. It is conveneint to evaluate the Jacobian matrix and other approximated values at the Gauss points to evaluate the integral directly using gaussian quadrature (note that quadrature points are derived for a one dimensional problem and the two dimensional extension is not necessarily valid).

Appendix

Mathematical Tools

Leibnitz rule:

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\mathrm{d}y + f[x,b(x)]\frac{\mathrm{d}b}{\mathrm{d}x} - f[x,a(x)]\frac{\mathrm{d}a}{\mathrm{d}x} \qquad (4.81)$$

Stoke's Equation:

$$\int_S (\nabla \times \vec{v})\mathrm{d}S = \oint_C \vec{v}\mathrm{d}C \qquad (4.82)$$

Divergence Theorem:

$$\int_S \vec{v}\cdot\vec{n}\mathrm{d}S = \int_V \nabla\cdot\vec{v}\mathrm{d}V \qquad (4.83)$$

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