VIGRE Nonlinear Dynamics Group Presentations

Week of June 5th, 2000
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These presentations follow the general outline presented in Strogatz, *Nonlinear Dynamics and Chaos*

1 Phase-Portraits and Linear Dynamical Systems (Smita Sihag)

1.1 Differential Equations and Fixed Points

Definition 1.1. A *fixed point* of a differential equation is a point such that $\dot{x} = 0$.

In other words, if the differential equation is interpreted as a vector field, then the flow is zero at a fixed point. Points where $\dot{x} \neq 0$ are also important. The flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. This helps us further classify fixed points according to their stability.

Definition 1.2. A *stable* fixed point is one towards which flow occurs (i.e. $\dot{x} > 0$ to its left and $\dot{x} < 0$ to its right. In contrast, the direction of flow is away from an *unstable* fixed point.

The signs of $\dot{x}$ in the neighborhood of an unstable fixed point are the reverse of those around a stable fixed point.

1.1.1 An Easy Example

Consider the nonlinear differential equation

$$\dot{x} = \sin x, \qquad 0 \leq x \leq 2\pi. \tag{1.1}$$

Differential equation (1.1) can be easily solved by the method of separation of variables:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \sin x \Rightarrow \mathrm{d}t = \frac{\mathrm{d}x}{\sin x} \Rightarrow t = \int \csc x \mathrm{d}x = -\ln|\csc x + \cot x| + \mathrm{C}$$

Suppose now that the initial condition is that $x = x_0$ at $t = 0$. Then solving for the constant C gives $\mathrm{C} = \ln|\csc x_0 + \cot x_0|$. Therefore the solution to (1.1) is:

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|. \tag{1.2}$$

In this case, the fixed points occur at $0, \pi$, and 2π. A graph of x versus $\dot{x}$ makes it clear that only π is a stable fixed point. If, for example, a trajectory begins at $\frac{\pi}{4}$, then that trajectory will asymptotically approach the stable fixed point π.

1.2 Two-Dimensional Linear Systems

Our goal is to find a solution to the general two-dimensional linear system of equations,

$$\begin{cases} \dot{x} = \mathrm{a}x + \mathrm{b}y \\ \dot{y} = \mathrm{c}x + \mathrm{d}y \end{cases} \tag{1.3}$$

The system of equations (1.3) can also be written in vector notation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Let's denote by **A** the above 2x2 matrix:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then, the linear system is equivalent to

$$\dot{\vec{x}} = \mathbf{A}\vec{x}, \tag{1.4}$$

where $\vec{x} = \binom{x}{y}$. Let us suppose that the solution to this system is given by

$$\vec{x} = \mathrm{e}^{\lambda t}\vec{v}, \tag{1.5}$$

where $\vec{v} = \binom{v_1}{v_2}$. Then, plugging equation (1.5) into (1.4) gives

$$\lambda \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\mathbf{A}\vec{v}. \tag{1.6}$$

Dividing both sides of (1.6) by $\mathrm{e}^{\lambda t}$ leaves us with the equation

$$\mathbf{A}\vec{v} = \lambda\vec{v}, \tag{1.7}$$

which means that $\vec{v}$ is an eigenvector for the eigenvalue λ. We then call (1.5) an eigensolution. This, therefore, is a good time to brush up on our linear algebra.

1.2.1 How to Find Eigenvalues and Eigenvectors

In order to find the eigenvalues of our matrix **A**, we must first posit a definition.

Definition 1.3. The *characteristic equation* of an nxn matrix **M** is

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0,$$

where **I** is the identity matrix.

For our 2x2 matrix $\mathbf{A}$, the characteristic equation is

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Taking the determinant gives

$$\lambda^2 - \tau\lambda + \Delta = 0, \tag{1.8}$$

where

$$\begin{aligned} \tau &= \text{trace}(\mathbf{A}) = a + d, \\ \Delta &= \det(\mathbf{A}) = ad - bc. \end{aligned}$$

Using the quadratic equation, we find that

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{1.9}$$

are the eigenvalues of equation (1.8) Assuming that the values of λ_1, λ_2 we found in (1.9) are distinct, we know that the corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. These eigenvectors are found by solving equation (1.7) with the appropriate values of λ plugged in. We then write down a general solution for our linear system:

$$\vec{x}(t) = \mathrm{c}_1 \mathrm{e}^{\lambda_1 t}\vec{v}_1 + \mathrm{c}_2 \mathrm{e}^{\lambda_2 t}\vec{v}_2 \tag{1.10}$$

1.3 Further Classification of Fixed Points

Consider $\tau^2 - 4\Delta = 0$, the discriminant of the characteristic equation (1.8). This discriminant allows one to classify fixed points in a very specific way. To better understand the following definitions, consider the graph of $\tau^2 - 4\Delta = 0$, drawn with Δ on the horizontal axis and τ on the vertical axis. This graph is a parabola which opens to the right:

Definition 1.4. A fixed point is a *saddle* if $\Delta < 0$.

Definition 1.5. A fixed point is a *stable node* if $\Delta = 0$.

Definition 1.6. A fixed point is an *unstable node* if $\tau = 0$.

Definition 1.7. A fixed point is a *star OR degenerate node* if $\tau^2 = 4\Delta$.

Definition 1.8. A fixed point is an *unstable spiral* if $0 < \tau < \sqrt{4\Delta}$.

Definition 1.9. A fixed point is a *stable spiral* if $-\sqrt{4\Delta} < \tau < 0$.

Spirals, (nondegenerate) nodes, and saddles are considered to be *hyperbolic* fixed points, which we will explore further in the next section.

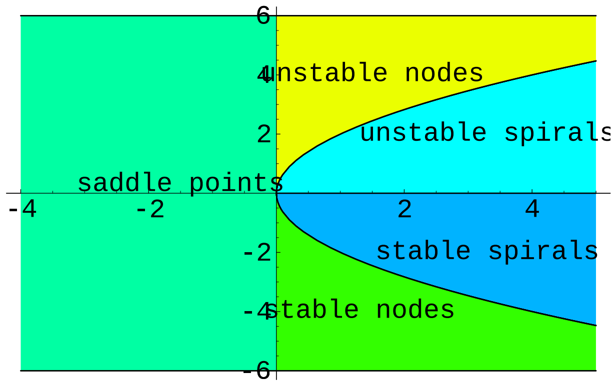

Figure 1: Figure courtesy of Andrew Rudman

2 The Linearization Technique for Nonlinear Systems (Andrew Rudman)

We consider a general nonlinear system of equations,

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{2.1}$$

and we suppose that (x^*, y^*) is a fixed point (meaning that $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$). Let

$$u = x - x^*, \quad v = y - y^*$$

denote a small disturbance from (x^*, y^*). Now, to see what happens to this disturbance, we derive differential equations for u and v. Because x^* is a constant,

$$\dot{u} = \dot{x}.$$

Then, by substituting into the system (2.1), we have that

$$\dot{u} = f(x^* + u, y^* + v),$$

which we can expand into its Taylor series:

$$\dot{u} = f(x^*, y^*) + u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*) + O(u^2, v^2, uv), \tag{2.2}$$

where $O(u^2, v^2, uv)$ denotes quadratic terms in u and v. These terms are very small because u and v are small. In the same manner, we derive the differential equation for v:

$$\dot{v} = g(x^*, y^*) + u\frac{\partial g}{\partial x}(x^*, y^*) + v\frac{\partial g}{\partial y}(x^*, y^*) + O(u^2, v^2, uv) \tag{2.3}$$

Then the evolution of the disturbance (u, v) can be expressed as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.} \tag{2.4}$$

Definition 2.1. We call the matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

the *Jacobian matrix* at the fixed point (x^*, y^*).

The quadratic terms in equation (2.4) are so small that it might be a good idea to throw them out altogether, in which case we would obtain a linearized system that can be analyzed more simply, such as by methods expounded in Section 1.

2.0.1 But Can We Neglect the Quadratic Terms?

It depends. If the fixed point is hyperbolic (as discussed in Section 1.3), then it is safe to do this. The linearized system does give a locally correct phase portrait around the fixed point if the fixed point for the linearized system is not one of the borderline cases–a center, a star or degenerate node, or a nonsingular fixed point (refer to Section 1.3 if necessary). So, if the linearized system predicts a saddle, node, or spiral, the fixed point is indeed a saddle, node, or spiral for the original nonlinear system. Therefore, hyperbolic fixed points are very important and it would be a good idea to get a better understanding of them.

2.1 Hyperbolic Fixed Points

Definition 2.2. A fixed point of an nth-order system is called *hyperbolic* if $\mathrm{Re}(\lambda_i) \neq 0 \ \forall \ i = 1, \ldots, n$.

There is also an important theorem concerning fixed points that we should know about, but first we must clarify what "topologically equivalent" means.

Definition 2.3. *Topologically equivalent* means that there is a homeomorphism (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time is preserved.

With this in mind, we can understand the famous *Hartman-Grobman theorem*:

Theorem 2.4. *The local phase portrait near a hyperbolic fixed point is topologically equivalent to the phase portrait of the linearization, and in fact the stability type of the fixed point is preserved by the linearization.*

2.2 A Counterexample: Why You Should Only Linearize in the Case of Hyperbolic Fixed Points

Let's examine the nonlinear system of equations

$$\begin{cases} \dot{x} = & -y + ax(x^2+y^2) \\ \dot{y} = & x + ay(x^2+y^2) \end{cases} \tag{2.5}$$

The fixed point of this system occurs at $(0,0)$. By a shortcut to linearization, we can eliminate the nonlinear terms to get the linearization

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

For the matrix A in (2.6), $\tau = 0$ and $\Delta = 1$. Therefore, the fixed point $(0,0)$ is a center, which is nonhyperbolic. To emphasize why linearization should not be used on this system, let us complete the process.

In this case it is elucidating to express system (2.5) in polar coordinates. We let $x = r\cos\Theta$ and $y = r\sin\Theta$. Then, $r^2 = x^2 + y^2$ and by the chain rule,

$$r\dot{r} = x\dot{x} + y\dot{y} \tag{2.7}$$

Plugging in the definitions of $\dot{x}$ and $\dot{y}$ from system (2.5) into equation (2.7), we find that

$$\begin{aligned} r\dot{r} &= x(y + ax(x^2+y^2)) + y(x + ay(x^2+y^2)) \\ &= -xy + ax^2(x^2+y^2) + xy + ay^2(x^2+y^2) \\ &= a(x^2+y^2)^2 \\ &= ar^4. \end{aligned}$$

Therefore,

$$\dot{r} = ar^3.$$

To find an expression for $\dot{\Theta}$, we use the substitution

$$\Theta = \tan^{-1}\left|\frac{y}{x}\right|$$

and find that

$$\dot{\Theta} = \frac{x\dot{y} - y\dot{x}}{r^2},$$

which, after plugging in for $\dot{x}$ and $\dot{y}$ from the system, yields

$$\dot{\Theta} = 1.$$

This means that system (2.5) can be written in polar coordinates as

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\Theta} = 1 \end{cases} \tag{2.8}$$

The system, as written in (2.8), makes the reason that we cannot linearize in this case clear. If $a = 0$, then the phase portrait consists of concentric circular trajectories moving in the counterclockwise direction and centered about the fixed point $(0, 0)$. However, if $a > 0$, then r is increasing and the phase portrait is a spiral coming out of the fixed point $(0, 0)$ (the origin is an unstable spiral). If $a < 0$, then r is decreasing and the phase portrait is a spiral coming into the fixed point $(0, 0)$ (the origin is a stable spiral). Trajectories are required to close perfectly after one cycle, but in this particular case (in which we have a nonhyperbolic fixed point), the trajectory is thrown off track into a spiral by the nonnegligible quadratic terms in the nonlinear system. The moral of the story is that we must ensure that the system has hyperbolic fixed points before we attempt to linearize it.

3 Conservative Systems (Martin Andersen)

If a particle of mass m is subjected to a nonlinear force $F(x)$, then by Newton's law $F = ma$, the equation of motion is

$$m\ddot{x} = F(x).$$

We make the claim that energy is conserved under the assumption that F is independent of both $\dot{x}$ and t. Let $V(x)$ be the potential energy, with $F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$. Substituting, we get

$$m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0. \tag{3.1}$$

Then, we solve differential equation (3.1) using the integrating factor $\dot{x}$:

$$m\ddot{x}\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow \int m\ddot{x}\dot{x} + \int \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \int 0 \Rightarrow \frac{1}{2}m\dot{x}^2 + V(x) = E, \tag{3.2}$$

where E is the total energy and is constant as a function of time.

Definition 3.1. A *conserved quantity* is a real valued continuous function $E(\vec{x})$ such that $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ and $E(\vec{x})$ is nonconstant on every open set. Then, we call a system for which a conserved quantity exists a *conservative system.*

3.1 An Example of a Conservative System

Suppose a particle of mass $m = 1$ is moving in a potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. We will now find and classify the fixed points for this system. Clearly, $\frac{\mathrm{d}V}{\mathrm{d}x} = -x + x^3$. Then, we rewrite our system as

$$\begin{cases} \dot{x} = & y \\ \dot{y} = & x - x^3 \end{cases} \tag{3.3}$$

The fixed points of system (3.3) occur at $(0,0)$ and $(\pm 1, 0)$ and the Jacobian is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}.$$

Evaluated at the fixed point $(0,0)$, the trace $\tau = 0$ and the determinant $\Delta = -1$. Therefore $(0,0)$ is a saddle. By symmetry at the fixed points $(\pm 1, 0)$, evaluated at either point $\tau = 0$ and $\Delta = 2$. Therefore $(\pm 1, 0)$ are centers. As it turns out, saddles and centers are very common types of fixed points for conservative systems. In consideration of our discussion in Section 2, we may worry that the small nonlinear terms could destroy the center predicted by linearization. But energy conservation saves the day here. The trajectories are closed curves defined by contours of constant energy:

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{constant}$$

In fact, we even have a theorem asserting this:

Theorem 3.2. *(Nonlinear centers for conservative systems) Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y) \in \mathrm{R}^2$, and $\boldsymbol{f}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\boldsymbol{x})$ and suppose that $\boldsymbol{x}^*$ is an isolated fixed point. If $\boldsymbol{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\boldsymbol{x}^*$ are closed.*

3.2 Of Particular Interest: Finding the Period of the Duffing Oscillator

Using conservation of energy, we will find the oscillation period of the Duffing Oscillator

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad 0 < \varepsilon \ll 1, \quad x(0) = a, \dot{x}(0) = 0. \tag{3.4}$$

Then,

$$E_0 = \frac{1}{2}(\dot{x})^2 + V(x), \quad V(x) = \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4. \tag{3.5}$$

Plugging the initial conditions from (3.4) into (3.5), we find that

$$E_0 = \frac{1}{2}(0)^2 + V(a) = \frac{1}{2}a^2 + \frac{\varepsilon}{4}a^4.$$

Therefore,

$$(\dot{x})^2 = a^2 - x^2 + \frac{\varepsilon}{2}(a^4 - x^4),$$

This means that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = [a^2 - x^2 + \frac{\varepsilon}{2}(a^4 - x^4)]^{\frac{1}{2}}.$$

Separating variables and integrating yields

$$\int_0^{T(\varepsilon)} \mathrm{d}t = 2\int_a^{-a} \frac{\mathrm{d}x}{\sqrt{a^2 - x^2 + \frac{\varepsilon}{2}(a^4 - x^4)}}. \tag{3.6}$$

Evaluating the integral on the left-hand side of (3.6) and taking the Taylor Series of the expression inside the integral on the right hand side, we find that

$$\begin{aligned} T(\varepsilon) &= [-\arctan(\frac{x\frac{1}{\sqrt{a^2-x^2}}}{x^2 - a^2}) + \frac{3}{8}a^2 \arctan(\frac{x\sqrt{a^2 - x^2}}{x^2 - a^2})\varepsilon \\ &- \frac{57}{256}a^4 \arctan(\frac{x\sqrt{a^2 - x^2}}{x^2 - a^2})\varepsilon^2 + O(\varepsilon^3)]_a^{-a} \end{aligned} \tag{3.7}$$

Simplifying (3.7) results in the expression

$$T(\varepsilon) = \pi - \frac{3}{8}\pi a^2 \varepsilon + \frac{57}{256}\pi a^4 \varepsilon^2 + O(\varepsilon^3). \tag{3.8}$$